2. $f(x,y = (\frac{2x}{y})^{xy}$

a) Allm. $D_{\hat{u}} f(a,b) = \nabla f(a,b) \cdot \hat{u}$

$\nabla f(a,b) = (f_1(a,b), f_2(a,b))$

$\hat{u} = \frac{u}{|u|}$

$$f(x,y) = \left(\frac{2x}{y}\right)^{xy} = e^{xy \cdot \ln\left(\frac{2x}{y}\right)}$$

$$f_1(x,y) = e^{xy\ln\left(\frac{2x}{y}\right)} \cdot \left(y \cdot \ln\left(\frac{2x}{y}\right) + xy \cdot \frac{1}{\frac{2x}{y}} \cdot \frac{2}{y}\right)$$

$$= \left(\frac{2x}{y}\right)^{xy} \cdot \left(y \cdot \ln\left(\frac{2x}{y}\right) + y\right) \Rightarrow f_1(1,2) = 2$$

$$f_2(x,y) = e^{xy\ln\left(\frac{2x}{y}\right)} \cdot \left(x \cdot \ln\left(\frac{2x}{y}\right) + xy \cdot \frac{1}{\frac{2x}{y}} \cdot \left(-\frac{2x}{y^2}\right)\right)$$

$$= \left(\frac{2x}{y}\right)^{xy} \cdot \left(x \cdot \ln\left(\frac{2x}{y}\right) - x\right) \Rightarrow f_2(1,2) = -1$$

$\therefore$ $\nabla f(1,2) = (2,-1)$

$u = (3,1) \Rightarrow \hat{u} = \frac{(3,1)}{\sqrt{10}} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$

$$D_{\hat{u}} f(1,2) = (2,-1) \cdot \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right) = \frac{6}{\sqrt{10}} - \frac{1}{\sqrt{10}} = \frac{5}{\sqrt{10}} = \frac{\sqrt{10}}{2}$$

b) Allm. Tangentplanets ekv.

$$z - f(a,b) = f_1(a,b) \cdot (x-a) + f_2(a,b) \cdot (y-b)$$

$f(1,2) = 1$, $f_1(1,2) = 2$, $f_2(1,2) = -1$

$\therefore$ $z - 1 = 2(x-1) - (y-2)$

$z - 1 = 2x - 2 - y + 2$

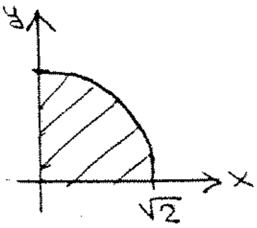
$2x - y - z + 1 = 0$

Svar: a) $D_{\hat{u}} f(1,2) = \frac{\sqrt{10}}{2}$

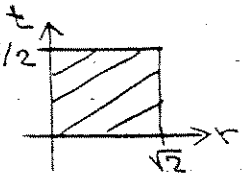
b) $2x - y - z + 1 = 0$

3. $\iint_D xy \ln(1+x^2+y^2)\,dx\,dy$

$D = \{(x,y) : x^2+y^2 \le 2,\ x \ge 0,\ y \ge 0\}$

D: 

Låt $\begin{cases} x = r\cos t, & 0 \le r \le \sqrt{2} \\ y = r\sin t, & 0 \le t \le \pi/2 \end{cases}$

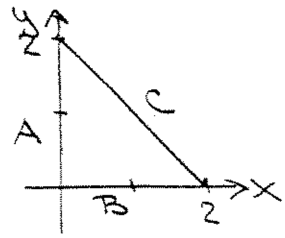
S: 

$\frac{\partial(x,y)}{\partial(r,t)} = r$

$$\therefore \iint_D xy \ln(1+x^2+y^2)\,dx\,dy =$$
$$\iint_S r\cos t \cdot r\sin t \cdot \ln(1+r^2)\cdot r\,dr\,dt =$$
$$\iint_S \frac{1}{2}\sin 2t \cdot r^3 \ln(1+r^2)\,dr\,dt =$$
$$\int_0^{\sqrt{2}} \left(\int_0^{\pi/2} \frac{1}{2}\sin 2t \cdot r^3 \ln(1+r^2)\,dt \right) dr =$$
$$\int_0^{\sqrt{2}} \left[-\frac{1}{4}\cos 2t \cdot r^3 \ln(1+r^2) \right]_{t=0}^{t=\pi/2} dr =$$
$$\int_0^{\sqrt{2}} \frac{1}{2} r^3 \ln(1+r^2)\,dr =$$
$$\frac{1}{2}\cdot\left(\left[\frac{r^4}{4}\cdot \ln(1+r^2) \right]_0^{\sqrt{2}} - \int_0^{\sqrt{2}} \frac{r^4}{4}\cdot\frac{2r}{1+r^2}\,dr \right) =$$
$$\frac{1}{2}\cdot\left(\ln 3 - \frac{1}{2}\int_0^{\sqrt{2}} \left(r^3 - r + \frac{r}{1+r^2}\right) dr \right) =$$
$$\frac{1}{2}\cdot\left(\ln 3 - \frac{1}{2}\cdot\left[\frac{r^4}{4} - \frac{r^2}{2} + \tfrac{1}{2}\ln(1+r^2) \right]_0^{\sqrt{2}} \right) =$$
$$\frac{1}{2}\cdot\left(\ln 3 - \frac{1}{2}\left(1 - 1 + \tfrac{1}{2}\ln 3\right)\right) =$$
$$\frac{1}{2}\cdot\left(\ln 3 - \frac{1}{4}\ln 3 \right) = \frac{3\ln 3}{8}$$

<u>Svar:</u> $\frac{3\ln 3}{8}$

4. $f(x,y) = (1-x^2+y^2)\,e^{x-y}$, $y \le 2-x$, $x \ge 0$, $y \ge 0$



f kont. på slutet begr. område $\Rightarrow$
största och minsta värde antas.

I. Inre punkter

$f_1(x,y) = -2x \cdot e^{x-y} + (1-x^2+y^2)e^{x-y} = e^{x-y}(1-2x-x^2+y^2)$

$f_2(x,y) = 2y \cdot e^{x-y} + (1-x^2+y^2)\cdot(-1)e^{x-y} = e^{x-y}(2y-1+x^2-y^2)$

$\left.\begin{array}{ll} f_1(x,y)=0: & 1-2x-x^2+y^2=0 \\ f_2(x,y)=0: & 2y-1+x^2-y^2=0 \end{array}\right\} \Rightarrow \quad y=x$

$\left.\begin{array}{l} y=x \\ f_1(x,y)=0 \end{array}\right\} \quad 1-2x-x^2+x^2=0 \quad \Rightarrow \quad x=1/2$

Kritisk punkt: $(1/2, 1/2)$

$f(1/2, 1/2) = 1$

II. Randpunkter

A. $x=0$, $0 \le y \le 2$

$f = (1+y^2)e^{-y} \quad \Rightarrow \quad f' = 2y\,e^{-y} - (1+y^2)e^{-y} = e^{-y}(2y-1-y^2)$

$f'=0$: $y^2-2y+1=0 \Rightarrow y = 1 \pm 0$

$f(0,0)=1$, $f(0,1) = 2e^{-1} \approx 0.74$, $f(0,2) = 5e^{-2} \approx 0.68$

B. $y=0$, $0 \le x \le 2$

$f = (1-x^2)e^x \quad \Rightarrow \quad f' = -2x\,e^x + (1-x^2)e^x = (1-2x-x^2)e^x$

$f'=0$: $x^2+2x-1=0 \Rightarrow x = -1 \oplus \sqrt{2}$

$f(0,0)=1$, $f(\sqrt{2}-1, 0) = 2(\sqrt{2}-1)e^{\sqrt{2}-1} \approx 1.25$, $f(2,0) = -3e^2 \approx -22$

C. $y = 2-x$, $0 \le x \le 2$

$f = (1-x^2+(2-x)^2)\,e^{x-(2-x)} = (5-4x)\,e^{2x-2}$

$f' = -4\,e^{2x-2} + (5-4x)\cdot 2\,e^{2x-2} = (6-8x)e^{2x-2}$

$f'=0$: $6-8x=0 \Rightarrow x = 3/4$

$f(0,2) = 5e^{-2}$, $f(3/4, 5/4) = 2e^{-1/2} \approx 1.21$, $f(2,0) = -3e^2$

<u>Svar:</u> Största värde $= 2(\sqrt{2}-1)e^{\sqrt{2}-1}$ i $(\sqrt{2}-1, 0)$
Minsta värde $= -3e^2$ i $(2,0)$

5. $C: \quad x = t\cos t \ , \quad y = t\sin t \ , \quad z = 2t$

$P_1 = (0,0,0) \quad \Rightarrow \quad t = 0$

$P_2 = (2\pi, 0, 4\pi) \Rightarrow \quad t = 2\pi$

$\therefore \quad \mathbf{r}(t) = (t\cos t, t\sin t, 2t) \ , \quad 0 \le t \le 2\pi$

$$\frac{d\mathbf{r}}{dt} = (\cos t - t\sin t, \sin t + t\cos t, 2)$$

$$\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2^2}$$

$$= \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos^2 t + 4}$$

$$= \sqrt{t^2 + 5}$$

a) <u>Allm.</u> $\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}) \cdot \left|\frac{d\mathbf{r}}{dt}\right| dt$

$$\therefore \int_C z\,ds = \int_0^{2\pi} 2t \cdot \sqrt{t^2+5}\,dt =$$

$$= \left| \begin{matrix} \sqrt{t^2+5} = p \\ t = \sqrt{p^2-5} \end{matrix} \quad \frac{dt}{dp} = \frac{p}{\sqrt{p^2-5}} \quad \begin{matrix} t = 2\pi \Rightarrow p = \sqrt{4\pi^2+5} \\ t = 0 \Rightarrow p = \sqrt{5} \end{matrix} \right|$$

$$= \int_{\sqrt{5}}^{\sqrt{4\pi^2+5}} 2\sqrt{p^2-5} \cdot p \cdot \frac{p}{\sqrt{p^2-5}}\,dp = \int_{\sqrt{5}}^{\sqrt{4\pi^2+5}} 2p^2\,dp =$$

$$= \left[\frac{2}{3}p^3\right]_{\sqrt{5}}^{\sqrt{4\pi^2+5}} = \frac{2}{3}\left((4\pi^2+5)\sqrt{4\pi^2+5} - 5\sqrt{5}\right)$$

b) <u>Allm.</u> $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{dt}\,dt$

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (2t^2\cos t, 2t^2\sin t, t^2\sin t\cos t) \cdot$$

$$\cdot (\cos t - t\sin t, \sin t + t\cos t, 2)\,dt$$

$$= \int_0^{2\pi} (2t^2\cos^2 t - 2t^3\sin t\cos t + 2t^2\sin^2 t + 2t^3\sin t\cos t + 2t^2\sin t\cos t)\,dt$$

$$= \int_0^{2\pi} (2t^2 + 2t^2\sin t\cos t)\,dt = \int_0^{2\pi} 2t^2\,dt + \int_0^{2\pi} t^2\sin 2t\,dt$$

$$= \left[\frac{2}{3}t^3\right]_0^{2\pi} + \left[t^2 \cdot \frac{(-\cos 2t)}{2}\right]_0^{2\pi} - \int_0^{2\pi} 2t \cdot \frac{(-\cos 2t)}{2}\,dt =$$

$$= \frac{16\pi^3}{3} + 4\pi^2 \cdot \left(-\frac{1}{2}\right) + \int_0^{2\pi} t\cos 2t\,dt =$$

$$= \frac{16\pi^3}{3} - 2\pi^2 + \left[t \cdot \frac{\sin 2t}{2}\right]_0^{2\pi} - \int_0^{2\pi} \frac{\sin 2t}{2}\,dt =$$

$$= \frac{16\pi^3}{3} - 2\pi^2 + \left[\frac{\cos 2t}{4}\right]_0^{2\pi} = \frac{16\pi^3}{3} - 2\pi^2$$

<u>Svar:</u> a) $\frac{2}{3}\left((4\pi^2+5)\sqrt{4\pi^2+5} - 5\sqrt{5}\right)$

b) $\frac{16\pi^3}{3} - 2\pi^2$

6. Givna plan: $x+y-z=3$, $x-y+z=1$

Låt "avståndsfunktionen" $f(x,y,z)=\sqrt{x^2+y^2+z^2}$

och bivillkorsfunktionerna vara

$g(x,y,z)=x+y-z-3$ och $h(x,y,z)=x-y+z-1$

Bilda nu Lagrangesfunktionen

$L(x,y,z,\lambda_1,\lambda_2)=(f(x,y,z))^2+\lambda_1\cdot g(x,y,z)+\lambda_2\cdot h(x,y,z)$:

$$L(x,y,z,\lambda_1,\lambda_2)=x^2+y^2+z^2+\lambda_1(x+y-z-3)+\lambda_2(x-y+z-1)$$

$$\begin{aligned}
L_1 &= 2x+\lambda_1+\lambda_2\\
L_2 &= 2y+\lambda_1-\lambda_2\\
L_3 &= 2z-\lambda_1+\lambda_2\\
L_4 &= x+y-z-3\\
L_5 &= x-y+z-1
\end{aligned}$$

$$\left.\begin{aligned}
L_1=0: &\quad 2x+\lambda_1+\lambda_2=0\\
L_2=0: &\quad 2y+\lambda_1-\lambda_2=0\\
L_3=0: &\quad 2z-\lambda_1+\lambda_2=0
\end{aligned}\right\}\Rightarrow y=-z$$
$$\begin{aligned}
L_4=0: &\quad x+y-z-3=0\\
L_5=0: &\quad x-y+z-1=0
\end{aligned}$$

$$\left.\begin{aligned}
&\left.\begin{aligned}y&=-z\\ L_4&=0\end{aligned}\right\}\quad x-2z-3=0\\
&\left.\begin{aligned}y&=-z\\ L_5&=0\end{aligned}\right\}\quad x+2z-1=0
\end{aligned}\right\}\Rightarrow 2x-4=0 \Rightarrow x=2$$

$$\left.\begin{aligned}x&=2\\ x+2z-1&=0\end{aligned}\right\}\Rightarrow 2+2z-1=0 \Rightarrow z=-1/2$$

$$\left.\begin{aligned}z&=-1/2\\ y&=-z\end{aligned}\right\}\Rightarrow y=1/2$$

$$\therefore\quad f(2,1/2,-1/2)=\sqrt{4+1/4+1/4}=\frac{\sqrt{18}}{2}=\frac{3\sqrt{2}}{2}$$

Detta är kortaste avståndet ty

Längsta avståndet "är" ∞.

<u>Svar:</u> $\frac{3\sqrt{2}}{2}$

7 a) $2z = 8 - x^2 - 2y^2 \quad , \quad z = \sqrt{x^2+2y^2}$

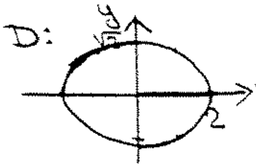
"Skärn" då $2\sqrt{x^2+2y^2} = 8 - x^2 - 2y^2 \Rightarrow$

$$x^2+2y^2+2\sqrt{x^2+2y^2}-8=0$$

Låt $x^2+2y^2=r^2$: $r^2+2r-8=0 \Rightarrow r=-1\pm 3 \ , r>0$

$\therefore$ Volymen $= \iint_D \left(4-\frac{1}{2}(x^2+2y^2)-\sqrt{x^2+2y^2}\right)dxdy$

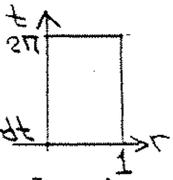
där $D=\{(x,y): x^2+2y^2\le 4\}$. $\frac{x^2}{4}+\frac{y^2}{2}\le 1$

D: 

Låt $\begin{cases} x = 2r\cos t & , \ 0\le r\le 1 \\ y=\sqrt{2}\, r\sin t & , \ 0\le t\le 2\pi \end{cases}$

$$\frac{\partial(x,y)}{\partial(r,t)} = \begin{vmatrix} 2\cos t & -2r\sin t \\ \sqrt{2}\sin t & \sqrt{2}r\cos t \end{vmatrix} = 2\sqrt{2}r\cos^2 t + 2\sqrt{2}r\sin^2 t = 2\sqrt{2}\,r$$

$\therefore$ Volymen $= \iint_D \left(4-\frac{1}{2}(x^2+2y^2)-\sqrt{x^2+2y^2}\right)dxdy =$

S: 

$= \iint_S \left(4-\frac{1}{2}(4r^2\cos^2 t+2\cdot 2r^2\sin^2 t)-\sqrt{4r^2\cos^2 t+2\cdot 2r^2\sin^2 t}\right)2\sqrt{2}\,r\,drdt$

$= 2\sqrt{2}\int_0^{2\pi}\left(\int_0^1 (4r-2r^3-2r^2)\,dr\right)dt = 2\sqrt{2}\int_0^{2\pi}\left[2r^2-\frac{r^4}{2}-\frac{2r^3}{3}\right]_{r=0}^{r=1}dt$

$= 2\sqrt{2}\int_0^{2\pi}\frac{5}{6}\,dt = \frac{5\sqrt{2}}{3}\cdot[t]_0^{2\pi} = \underline{\frac{10\pi\sqrt{2}}{3}}$

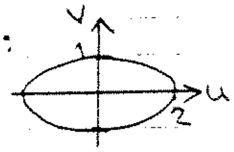
b) Ytan $z = 4-\frac{x^2}{2}-y^2 \quad , \quad x^2+4y^2\le 4$

Parameterframst. $\begin{cases} x=u \\ y=v \\ z=4-\frac{u^2}{2}-v^2 & , \quad u^2+4v^2\le 4 \end{cases}$

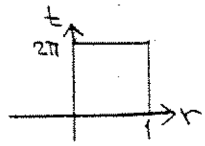
$\therefore \quad \mathbb{r}(u,v) = (u, v, 4-\frac{u^2}{2}-v^2) \quad , \quad \frac{u^2}{4}+v^2\le 1$

$$\frac{d\mathbb{r}}{du}\times\frac{d\mathbb{r}}{dv} = \begin{vmatrix} \dot{\imath} & \dot{\jmath} & \mathbb{k} \\ 1 & 0 & -u \\ 0 & 1 & -2v \end{vmatrix} = (u, 2v, 1)$$

$$\left|\frac{d\mathbb{r}}{du}\times\frac{d\mathbb{r}}{dv}\right| = \sqrt{u^2+4v^2+1}$$

D: 

$\therefore$ Arean $= \iint_D 1\cdot\sqrt{u^2+4v^2+1}\;du\,dv$

S: 

Låt $\begin{cases} u = 2r\cos t & , \ 0\le r\le 1 \\ v = r\sin t & , \ 0\le t\le 2\pi \end{cases}$

$$\frac{\partial(u,v)}{\partial(r,t)} = \begin{vmatrix} 2\cos t & -2r\sin t \\ \sin t & r\cos t \end{vmatrix} = 2r\cos^2 t + 2r\sin^2 t = 2r$$

$\therefore$ Arean $= \iint_D \sqrt{u^2+4v^2+1}\;dudv = \iint_S \sqrt{4r^2\cos^2 t+4r^2\sin^2 t+1}\cdot 2r\,drdt$

$= \iint_S \sqrt{4r^2+1}\cdot 2r\,drdt = \int_0^{2\pi}\left(\int_0^1 \sqrt{4r^2+1}\cdot 2r\,dr\right)dt =$

$= \int_0^{2\pi}\left[\frac{1}{6}(4r^2+1)^{3/2}\right]_{r=0}^{r=1}dt = \frac{1}{6}(5\sqrt{5}-1)\cdot\int_0^{2\pi}dt = \underline{\frac{\pi(5\sqrt{5}-1)}{3}}$

8. $\mathbb{F} = \left(ye^{xy} + ze^{x},\ xe^{xy} + \sqrt{\frac{z}{y}},\ e^{x} + \sqrt{\frac{y}{z}} \right)$

Konservativt vektorfält om $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \mathbf{0}$

dvs om $\frac{\partial}{\partial y} F_3 = \frac{\partial}{\partial z} F_2$, $\frac{\partial}{\partial x} F_3 = \frac{\partial}{\partial z} F_1$, $\frac{\partial}{\partial x} F_2 = \frac{\partial}{\partial y} F_1$

$\frac{\partial}{\partial y} F_3 = \frac{1}{2\sqrt{yz}}$ $\qquad \frac{\partial}{\partial x} F_3 = e^{x}$ $\qquad \frac{\partial}{\partial x} F_2 = e^{xy} + xye^{xy}$

$\frac{\partial}{\partial z} F_2 = \frac{1}{2\sqrt{yz}}$ $\qquad \frac{\partial}{\partial z} F_1 = e^{x}$ $\qquad \frac{\partial}{\partial y} F_1 = e^{xy} + xye^{xy}$

$\therefore$ Konservativt vektorfält

Vid konservativt vektorfält är $\int_C \mathbb{F} \cdot d\mathbf{r}$ oberoende av vägen dvs $\int_C \mathbb{F} \cdot d\mathbf{r} = \left[\phi(x,y,z) \right]_{P_1}^{P_2}$.

Sök nu $\phi(x,y,z)$ så $\nabla \phi(x,y,z) = (F_1, F_2, F_3)$:

$$\frac{\partial \phi}{\partial x} = ye^{xy} + ze^{x} \Rightarrow \phi(x,y,z) = \int (ye^{xy} + ze^{x})\, dx$$

$\therefore \quad \phi(x,y,z) = e^{xy} + ze^{x} + C_1(y,z)$

Derivera map y: $\frac{\partial \phi}{\partial y} = x \cdot e^{xy} + \frac{\partial C_1(y,z)}{\partial y}$

Men $\frac{\partial \phi}{\partial y} = x \cdot e^{xy} + \sqrt{\frac{z}{y}}$ $\Bigg\} \Rightarrow$

$$\frac{\partial C_1(y,z)}{\partial y} = \sqrt{\frac{z}{y}} = \sqrt{z} \cdot y^{-1/2} \Rightarrow C_1(y,z) = \int \sqrt{z} \cdot y^{-1/2}\, dy$$

$\therefore \quad C_1(y,z) = \sqrt{z} \cdot 2\sqrt{y} + C_2(z)$

$\therefore \quad \phi(x,y,z) = e^{xy} + ze^{x} + 2\sqrt{yz} + C_2(z)$

Derivera map z: $\frac{\partial \phi}{\partial z} = e^{x} + \sqrt{\frac{y}{z}} + C_2'(z)$

Men $\frac{\partial \phi}{\partial z} = e^{x} + \sqrt{\frac{y}{z}}$ $\Bigg\} \Rightarrow$

$$C_2'(z) = 0 \Rightarrow C_2(z) = C$$

$\therefore \quad \phi(x,y,z) = e^{xy} + ze^{x} + 2\sqrt{yz} + C$

$\mathbf{r}(t) = (\ln t, t, 2t)$, $1 \leq t \leq 2$

$t = 1$ ger $P_1 = (0, 1, 2)$
$t = 2$ ger $P_2 = (\ln 2, 2, 4)$

$$\therefore \int_C \mathbb{F} \cdot d\mathbf{r} = \left[e^{xy} + ze^{x} + 2\sqrt{yz} \right]_{(0,1,2)}^{(\ln 2, 2, 4)} =$$

$$= e^{2\ln 2} + 4 \cdot e^{\ln 2} + 2\sqrt{8} - (1 + 2 + 2\sqrt{2}) =$$

$$= 4 + 8 + 4\sqrt{2} - 3 - 2\sqrt{2} = 9 + 2\sqrt{2}$$