2. $f(x,y) = \sqrt{x(x+y)} = \sqrt{x^2+xy}$

a) $D_{\hat{u}} f(1,3) = \nabla f(1,3) \cdot \hat{u}$, $\nabla f(1,3) = (f_1(1,3), f_2(1,3))$, $\hat{u} = \frac{u}{|u|}$

$f_1(x,y) = \frac{2x+y}{2\sqrt{x^2+xy}}$, $f_1(1,3) = 5/4$

$f_2(x,y) = \frac{x}{2\sqrt{x^2+xy}}$, $f_2(1,3) = 1/4$

$\therefore \nabla f(1,3) = (5/4, 1/4)$

$u = (1,-1) \Rightarrow \hat{u} = \frac{(1,-1)}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

$\therefore D_{\hat{u}} f(1,3) = (5/4, 1/4) \cdot \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{5}{4\sqrt{2}} - \frac{1}{4\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

b) Maximala riktningsderivatan =

$|\nabla f(1,3)| = \sqrt{(5/4)^2 + (1/4)^2} = \frac{\sqrt{26}}{4}$

c) Tangentplanets ekv.: $z - f(a,b) = f_1(a,b)\cdot(x-a) + f_2(a,b)\cdot(y-b)$

$f(1,3) = 2$, $f_1(1,3) = 5/4$, $f_2(1,3) = 1/4$ ger

$$z - 2 = 5/4\,(x-1) + 1/4\,(y-3)$$

$$4z - 8 = 5x - 5 + y - 3$$

$$5x + y - 4z = 0$$

d) Normallinjens ekv. $\begin{cases} x = a + f_1(a,b)\cdot t \\ y = b + f_2(a,b)\cdot t \\ z = f(a,b) - t \end{cases}$, $t \in \mathbb{R}$

$f(1,3) = 2$, $f_1(1,3) = 5/4$, $f_2(1,3) = 1/4$ ger

$\begin{cases} x = 1 + 5/4\,t \\ y = 3 + 1/4\,t \\ z = 2 - t \end{cases}$, $t \in \mathbb{R}$ / $t = 4s$ ger / $\begin{cases} x = 1 + 5s \\ y = 3 + s \\ z = 2 - 4s \end{cases}$, $s \in \mathbb{R}$

Svar: a) $\frac{\sqrt{2}}{2}$

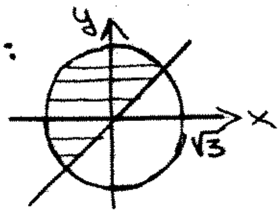
b) $\frac{\sqrt{26}}{4}$

c) $5x + y - 4z = 0$

d) $\begin{cases} x = 1 + 5s \\ y = 3 + s \\ z = 2 - 4s \end{cases}$, $s \in \mathbb{R}$

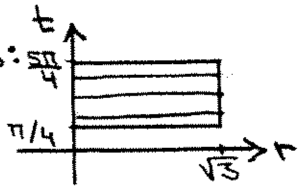
3. $\iint\limits_D x^2\sqrt{4-x^2-y^2}\,dx\,dy$

$D=\{(x,y):\ x^2+y^2\le 3\ ,\ x-y\le 0\}$

D: 

Låt $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$; $0\le r\le\sqrt{3}$; $\pi/4\le t\le 5\pi/4$

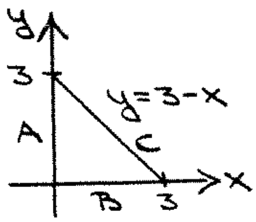
$\frac{\partial(x,y)}{\partial(r,t)} = r$

S: 

$$\therefore\ \iint\limits_D x^2\sqrt{4-x^2-y^2}\,dx\,dy =$$
$$=\iint\limits_S r^2\cos^2 t\cdot\sqrt{4-r^2\cos^2 t-r^2\sin^2 t}\cdot r\,dr\,dt =$$
$$=\iint\limits_S \cos^2 t\cdot r^3\sqrt{4-r^2}\,dr\,dt =$$
$$=\int_0^{\sqrt{3}} r^3\sqrt{4-r^2}\,dr\int_{\pi/4}^{5\pi/4}\cos^2 t\,dt =$$
$$=\int_0^{\sqrt{3}} r^3\sqrt{4-r^2}\,dr\int_{\pi/4}^{\cancel{5\pi}\,\pi/4}\frac{1+\cos 2t}{2}\,dt =$$
$$=\int_0^{\sqrt{3}} r^3\sqrt{4-r^2}\,dr\left[\frac{t}{2}+\frac{\sin 2t}{4}\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} =$$
$$=\int_0^{\sqrt{3}} r^3\sqrt{4-r^2}\,dr\left(\frac{5\pi}{8}+\frac{\sin\frac{5\pi}{2}}{4}-\left(\frac{\pi}{8}+\frac{\sin\frac{\pi}{2}}{4}\right)\right) =$$
$$=\frac{\pi}{2}\int_0^{\sqrt{3}} r^3\sqrt{4-r^2}\,dr = \left/\ \begin{matrix}\sqrt{4-r^2}=p & \frac{dr}{dp}=\frac{-p}{\sqrt{4-p^2}} & r=\sqrt{3}\Rightarrow p=1\\ r=\sqrt{4-p^2} & & r=0\Rightarrow p=2\end{matrix}\right.$$
$$=\frac{\pi}{2}\int_2^1 (4-p^2)\sqrt{4-p^2}\cdot p\cdot\frac{-p}{\sqrt{4-p^2}}\,dp =$$
$$=\frac{\pi}{2}\int_2^1 (p^4-4p^2)\,dp =$$
$$=\frac{\pi}{2}\cdot\left[\frac{p^5}{5}-\frac{4p^3}{3}\right]_2^1 =$$
$$=\frac{\pi}{2}\left(\frac{1}{5}-\frac{4}{3}-\left(\frac{32}{5}-\frac{32}{3}\right)\right) =$$
$$=\frac{\pi}{2}\left(\frac{28}{3}-\frac{31}{5}\right)=\frac{47\pi}{30}$$

Svar: $\frac{47\pi}{30}$

4. $f(x,y) = x^2 - 2xy + 2y^2 - 2y$



f kont. på slutet begr. område $\Rightarrow$ största och minsta värde antas.

I. Inre punkter

$f_1(x,y) = 2x - 2y = 2(x-y)$

$f_2(x,y) = -2x + 4y - 2$

$f_1(x,y) = 0 : \quad 2(x-y) = 0 \Rightarrow y = x$

$\left.\begin{array}{l} y = x \\ f_2(x,y) = 0 \end{array}\right\} \quad -2x + 4x - 2 = 0 \Rightarrow x = 1$

Kritisk punkt: $(1,1)$

$f(1,1) = 1 - 2 + 2 - 2 = -1$

II. Randpunkter

A. $x = 0 \quad , \quad 0 \le y \le 3$

$f = 2y^2 - 2y \Rightarrow f' = 4y - 2$. $f' = 0$ ger $y = 1/2$

$f(0,0) = 0 \quad , \quad f(0, 1/2) = -1/2 \quad , \quad f(0,3) = 12$

B. $y = 0 \quad , \quad 0 \le x \le 3$

$f = x^2 \Rightarrow f' = 2x$. $f' = 0$ ger $x = 0$

$f(0,0) = 0 \quad , \quad f(3,0) = 9$

C. $y = 3 - x \quad , \quad 0 \le x \le 3$

$f = x^2 - 2x(3-x) + 2(3-x)^2 - 2(3-x)$

$f = x^2 - 6x + 2x^2 + 18 - 12x + 2x^2 - 6 + 2x$

$f = 5x^2 - 16x + 12 \Rightarrow f' = 10x - 16$. $f' = 0$ ger $x = 8/5$

$f(0,3) = 12 \quad , \quad f(8/5, 7/5) = -4/5 \quad , \quad f(3,0) = 9$

Svar: Största värde $= 12$ i $(0,3)$

Minsta värde $= -1$ i $(1,1)$

5. $\int_C \mathbb{F}\cdot d\mathbb{r} = \int_C \mathbb{F}(\mathbb{r})\cdot \frac{d\mathbb{r}}{dt}\,dt$

$\mathbb{F} = (2y^3 - x^2y,\ x^3 - 2xy^2)$

$C:\quad x^2 + 4y^2 = 4 \quad , \quad y \geq 0 \ . \quad (2,0) \rightarrow (-2,0)$

$\frac{x^2}{4} + y^2 = 1$

Låt $\begin{cases} x = 2\cos t \\ y = \sin t \end{cases} \qquad 0 \leq t \leq \pi$

$\mathbb{r}(t) = (2\cos t, \sin t)$

$\frac{d\mathbb{r}}{dt} = (-2\sin t, \cos t)$

$\mathbb{F}(\mathbb{r}) = (2\sin^3 t - 4\cos^2 t\cdot \sin t,\ 8\cos^3 t - 4\cos t\cdot \sin^2 t)$

$\therefore \int_C \mathbb{F}\cdot d\mathbb{r} = \int_C \mathbb{F}(\mathbb{r})\cdot \frac{d\mathbb{r}}{dt}\,dt =$

$= \int_0^\pi (2\sin^3 t - 4\cos^2 t\cdot \sin t, 8\cos^3 t - 4\cos t\cdot \sin^2 t)\cdot(-2\sin t, \cos t)\,dt =$

$= \int_0^\pi (-4\sin^4 t + 8\cos^2 t\cdot \sin^2 t + 8\cos^4 t - 4\cos^2 t\cdot \sin^2 t)\,dt =$

$= \int_0^\pi (8\cos^2 t(\sin^2 t + \cos^2 t) - 4\sin^2 t(\sin^2 t + \cos^2 t))\,dt =$

$= \int_0^\pi (8\cos^2 t - 4\sin^2 t)\,dt =$

$= \int_0^\pi (8\cos^2 t - 4(1 - \cos^2 t))\,dt =$

$= \int_0^\pi (12\cos^2 t - 4)\,dt = \int_0^\pi (12\cdot \frac{1+\cos 2t}{2} - 4)\,dt =$

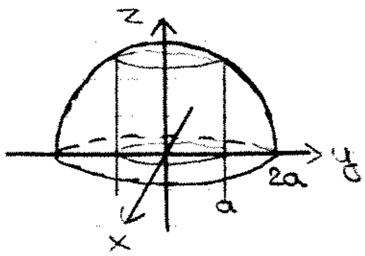
$= \int_0^\pi (2 + 6\cos 2t)\,dt = [2t + 3\sin 2t]_0^\pi =$

$= 2\pi$

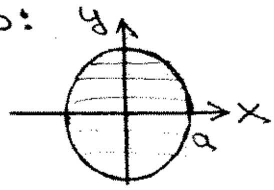
Svar: 2π

6. a) $x^2+y^2+z^2=4a^2$, $z \geq 0$

$x^2+y^2=a^2$

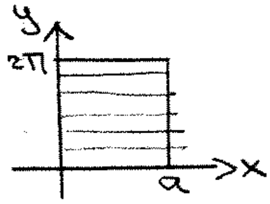


$$\text{Volymen} = \iiint_\Omega 1 \, dx\,dy\,dz =$$
$$= \iint_D \left(\int_0^{\sqrt{4a^2-x^2-y^2}} 1 \, dz \right) dx\,dy =$$
$$= \iint_D \left[z \right]_0^{\sqrt{4a^2-x^2-y^2}} dx\,dy =$$
$$= \iint_D \sqrt{4a^2-x^2-y^2} \, dx\,dy$$

D: 

Låt $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$ $\quad 0 \leq r \leq a$, $\; 0 \leq t \leq 2\pi$

$$\frac{\partial(x,y)}{\partial(r,t)} = r$$

S: 

$$\therefore \iint_D \sqrt{4a^2-x^2-y^2} \, dx\,dy =$$
$$= \iint_S \sqrt{4a^2-r^2\cos^2 t-r^2\sin^2 t} \cdot r \, dr\,dt =$$
$$= \iint_S \sqrt{4a^2-r^2} \cdot r \, dr\,dt = \int_0^a \sqrt{4a^2-r^2} \cdot r \, dr \int_0^{2\pi} dt =$$
$$= \int_0^a \sqrt{4a^2-r^2} \cdot r \, dr \, \left[t \right]_0^{2\pi} = 2\pi \int_0^a \sqrt{4a^2-r^2} \cdot r \, dr =$$
$$= 2\pi \left[-\frac{1}{3}(4a^2-r^2)^{3/2} \right]_0^a =$$
$$= 2\pi \left(-\frac{1}{3} \cdot 3a^2 \cdot \sqrt{3}a + \frac{1}{3} \cdot 4a^2 \cdot 2a \right) =$$
$$= \frac{2\pi(8-3\sqrt{3})a^3}{3}$$

6 b) $\quad x^2+y^2+z^2=4a^2 \quad , \quad x^2+y^2=a^2$

Låt $\quad x=u$
$\qquad y=v$
$\qquad z=\sqrt{4a^2-u^2-v^2} \quad , \quad u^2+v^2=a^2$

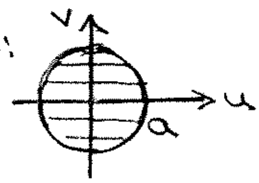
$$\mathbf{r}(u,v)=(u,v,\sqrt{4a^2-u^2-v^2})$$

$$\frac{d\mathbf{r}}{du}\times\frac{d\mathbf{r}}{dv}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & \frac{-u}{\sqrt{4a^2-u^2-v^2}} \\ 0 & 1 & \frac{-v}{\sqrt{4a^2-u^2-v^2}} \end{vmatrix}=\left(\frac{u}{\sqrt{4a^2-u^2-v^2}},\frac{v}{\sqrt{4a^2-u^2-v^2}},1\right)$$

$$\left|\frac{d\mathbf{r}}{du}\times\frac{d\mathbf{r}}{dv}\right|=\sqrt{\frac{u^2}{4a^2-u^2-v^2}+\frac{v^2}{4a^2-u^2-v^2}+1}=\frac{2a}{\sqrt{4a^2-u^2-v^2}}$$

$$\text{Arean}=\iint_D 1\cdot\frac{2a}{\sqrt{4a^2-u^2-v^2}}\,du\,dv=$$

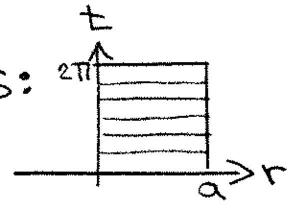
D:



Låt $\quad u=r\cos t \quad , \quad 0\le r\le a$
$\qquad v=r\sin t \quad ; \quad 0\le t\le 2\pi$

$$\frac{\partial(u,v)}{\partial(r,t)}=r$$

S:



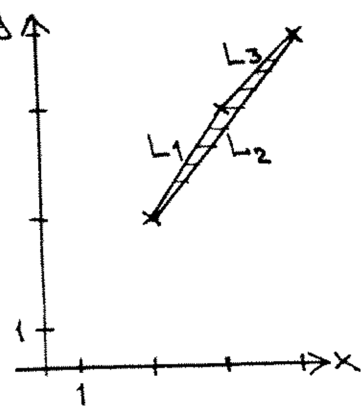
$$\therefore \quad \iint_D \frac{2a}{\sqrt{4a^2-u^2-v^2}}\,du\,dv=$$
$$=\iint_S \frac{2a}{\sqrt{4a^2-r^2}}\cdot r\;dr\,dt=$$
$$=\int_0^a \frac{2a}{\sqrt{4a^2-r^2}}\cdot r\,dr\int_0^{2\pi}dt=$$
$$=4\pi a\int_0^a\frac{r}{\sqrt{4a^2-r^2}}\,dr=4\pi a\left[-(4a^2-r^2)^{1/2}\right]_0^a$$
$$=4\pi a(-\sqrt{3}a+2a)=4\pi(2-\sqrt{3})a^2$$

Svar: $\quad$ a) $\dfrac{2\pi(8-3\sqrt{3})a^3}{3}$

$\qquad$ b) $4\pi(2-\sqrt{3})a^2$

8. $\displaystyle\iint_T \frac{(2(y-x)-1)^3}{(3x-2y)^2}\,dx\,dy$

T:



L_1: $(3,4)$, $(5,7)$. $k = 3/2$

$y-4 = 3/2\,(x-3) \iff 3x-2y=1$

L_2: $(3,4)$, $(7,9)$. $k = 5/4$

$y-4 = 5/4\,(x-3) \iff 5x-4y=-1$

L_3: $(5,7)$, $(7,9)$. $k = 1$

$y-7 = 1(x-5) \iff y-x=2$

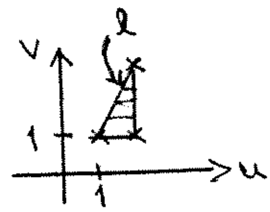
Låt $\begin{cases} u = y-x \\ v = 3x-2y \end{cases} \iff \begin{cases} x = 2u+v \\ y = 3u+v \end{cases}$

(x,y)	(u,v)
$(3,4)$	$(1,1)$
$(5,7)$	$(2,1)$
$(7,9)$	$(2,3)$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = 2-3 = -1$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1$$

D:



l: $v-1 = 2(u-1)$

$v = 2u-1$

$$\therefore \iint_T \frac{(2(y-x)-1)^3}{(3x-2y)^2}\,dx\,dy =$$

$$= \iint_D \frac{(2u-1)^3}{v^2}\cdot 1\,du\,dv =$$

$$= \int_1^2 \left(\int_1^{2u-1} \frac{(2u-1)^3}{v^2}\,dv \right) du =$$

$$= \int_1^2 \left[-\frac{(2u-1)^3}{v} \right]_{v=1}^{v=2u-1} du =$$

$$= \int_1^2 \left(-(2u-1)^2 + (2u-1)^3 \right) du =$$

$$= \left[-\frac{1}{6}(2u-1)^3 + \frac{1}{8}(2u-1)^4 \right]_1^2 =$$

$$= -\frac{1}{6}\cdot 3^3 + \frac{1}{8}\cdot 3^4 - \left(-\frac{1}{6}\cdot 1 + \frac{1}{8}\cdot 1\right) =$$

$$= -\frac{26}{6} + \frac{80}{8} = 10 - \frac{13}{3} = \frac{17}{3}$$

Svar: $\dfrac{17}{3}$