2. $f(x,y) = y e^{x(x-2y)}$

a) $D_{\hat{u}} f(a,b) = \nabla f(a,b) \cdot \hat{u}$ där

$\nabla f(a,b) = (f_1(a,b), f_2(a,b))$ och $\hat{u} = \frac{u}{|u|}$

$f(x,y) = y e^{x(x-2y)} = y e^{x^2-2xy}$

$f_1(x,y) = y e^{x^2-2xy} \cdot (2x-2y)$, $f_1(2,1) = 1 \cdot e^{4-4} \cdot (4-2) = 2$

$f_2(x,y) = e^{x^2-2xy} + y \cdot e^{x^2-2xy} \cdot (-2x)$, $f_2(2,1) = e^{4-4} + e^{4-4} \cdot (-4) = -3$

$\nabla f(2,1) = (2,-3)$

$u = (2,1) \Rightarrow \hat{u} = \frac{(2,1)}{\sqrt{4+1}} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$

$\therefore D_{\hat{u}} f(2,1) = (2,-3) \cdot \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = \frac{4}{\sqrt{5}} - \frac{3}{\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

b) Tangentplanets ekvation

$z - f(a,b) = f_1(a,b) \cdot (x-a) + f_2(a,b) \cdot (y-b)$

$f(2,1) = 1$, $f_1(2,1) = 2$, $f_2(2,1) = -3$ ger

$z - 1 = 2(x-2) - 3(y-1)$

$z - 1 = 2x - 4 - 3y + 3$

$2x - 3y - z = 0$

<u>Svar:</u> a) $\frac{\sqrt{5}}{5}$

b) $2x - 3y - z = 0$

3. $\int_C \mathbb{F} \cdot d\mathbb{r} = \int_a^b \mathbb{F}(\mathbb{r}(t)) \cdot \frac{d\mathbb{r}}{dt}\, dt$

$\mathbb{F} = (yz, xz, xy)$

C: Linjeus ekv. $\begin{cases} x = 2t \\ y = -t \\ z = -3t \end{cases}$, $0 \le t \le 1$

$\mathbb{r}(t) = (2t, -t, -3t)$, $0 \le t \le 1$

$\frac{d\mathbb{r}}{dt} = (2, -1, -3)$

$\mathbb{F}(\mathbb{r}(t)) = ((-t)\cdot(-3t), 2t\cdot(-3t), 2t\cdot(-t)) =$

$= (3t^2, -6t^2, -2t^2)$

$\therefore \int_C \mathbb{F} \cdot d\mathbb{r} = \int_0^1 (3t^2, -6t^2, -2t^2) \cdot (2, -1, -3)\, dt =$

$= \int_0^1 (6t^2 + 6t^2 + 6t^2)\, dt =$

$= \int_0^1 18t^2\, dt = [6t^3]_0^1 = 6$

<u>Svar:</u> $\int_C \mathbb{F} \cdot d\mathbb{r} = 6$

4. $y = a \cdot e^{bx}$

x	1	2	3	4	5
y	1.80	2.85	4.20	6.35	9.80

$$y = a \cdot e^{bx} \Rightarrow \ln y = \ln(a \cdot e^{bx}) \Rightarrow$$

$$\ln y = \ln a + \ln e^{bx} \Rightarrow \ln y = \ln a + bx$$

Normalekvationerna

$$\begin{pmatrix} \Sigma 1 & \Sigma x_k \\ \Sigma x_k & \Sigma x_k^2 \end{pmatrix} \begin{pmatrix} \ln a \\ b \end{pmatrix} = \begin{pmatrix} \Sigma \ln y_k \\ \Sigma x_k \cdot \ln y_k \end{pmatrix}$$

Nu $\Sigma 1 = 5$, $\Sigma x_k = 15$, $\Sigma x_k^2 = 55$

$\Sigma \ln y_k = 7.201027\ldots^{Q}$, $\Sigma x_k \cdot \ln y_k = 25.793409\ldots^{W}$

$$\left(\begin{array}{cc|c} 5 & 15 & 7.201027\ldots^{Q} \\ 15 & 55 & 25.793409\ldots^{W} \end{array}\right) \Leftrightarrow \left(\begin{array}{cc|c} 1 & 3 & 1.440205\ldots^{R} \\ 3 & 11 & 5.158681\ldots^{T} \end{array}\right) \Leftrightarrow$$

$$\left(\begin{array}{cc|c} 1 & 3 & 1.440205\ldots^{R} \\ 0 & 2 & 0.838065\ldots^{U} \end{array}\right) \Leftrightarrow \left(\begin{array}{cc|c} 1 & 3 & 1.440205\ldots^{R} \\ 0 & 1 & 0.419032\ldots^{P} \end{array}\right) \Leftrightarrow$$

$$\left(\begin{array}{cc|c} 1 & 0 & 0.183107\ldots^{O} \\ 0 & 1 & 0.419032\ldots^{P} \end{array}\right) \Rightarrow \begin{array}{l} \ln a = 0.183107\ldots \Rightarrow a \approx 1.20 \\ b \approx 0.419 \end{array}$$

$\therefore$ $a \approx 1.20$, $b \approx 0.419$ $\Rightarrow$ $y = 1.20 \cdot e^{0.419x}$

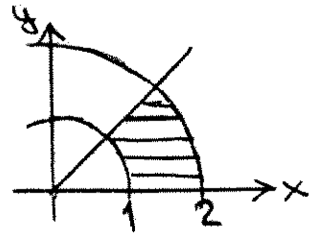
<u>Svar:</u> $a \approx 1.20$, $b \approx 0.419$

$y = 1.20 \cdot e^{0.419x}$

5. $\iint\limits_D (x^2-y^2)\, e^{x^2+y^2}\, dx\,dy$

$D = \{(x,y):\ 1 \le x^2+y^2 \le 4\ ,\ 0 \le y \le x\}$

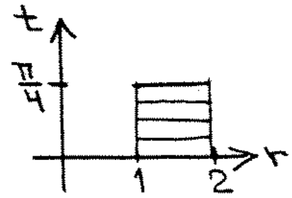
D:



Låt $\begin{cases} x = r\cos t \\ y = r\sin t \end{cases}$, $\begin{matrix} 1 \le r \le 2 \\ 0 \le t \le \pi/4 \end{matrix}$

$\frac{\partial(x,y)}{\partial(r,t)} = r$

S:



$\iint\limits_D (x^2-y^2)\cdot e^{x^2+y^2}\, dx\,dy =$

$\iint\limits_S (r^2\cos^2 t - r^2\sin^2 t)\cdot e^{r^2\cos^2 t + r^2\sin^2 t}\cdot r\ dr\,dt =$

$\iint\limits_S r^3(\cos^2 t - \sin^2 t)\cdot e^{r^2}\ dr\,dt =$

$\iint\limits_S r^3 e^{r^2}\cdot \cos 2t\ dr\,dt =$

$\int_1^2 \left(\int_0^{\pi/4} r^3 e^{r^2}\cos 2t\ dt \right) dr =$

$\int_1^2 \left[r^3 e^{r^2}\cdot \frac{\sin 2t}{2} \right]_{t=0}^{t=\pi/4} dr =$

$\int_1^2 r^3 e^{r^2}\cdot \frac{1}{2}\, dr = \frac{1}{2}\int_1^2 r^3 e^{r^2}\, dr = \left/ \begin{matrix} r^2 = p \\ r = \sqrt{p} \end{matrix}\quad \frac{dr}{dp} = \frac{1}{2\sqrt{p}}\quad \begin{matrix} r=2 \Rightarrow p=4 \\ r=1 \Rightarrow p=1 \end{matrix} \right/$

$\frac{1}{2}\int_1^4 p\sqrt{p}\, e^p \cdot \frac{1}{2\sqrt{p}}\, dp =$

$\frac{1}{4}\int_1^4 p\, e^p\, dp = \frac{1}{4}\left(\left[p e^p \right]_1^4 - \int_1^4 e^p\, dp \right) =$

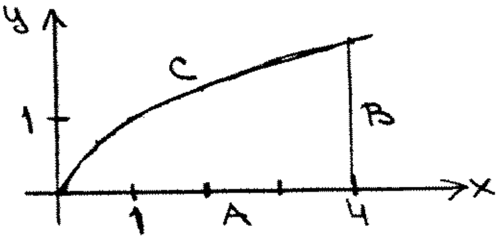
$\frac{1}{4}\left(\left[p e^p - e^p \right]_1^4 \right) =$

$\frac{1}{4}\left(4e^4 - e^4 - (e - e) \right) = \frac{3e^4}{4}$

<u>Svar:</u> $\frac{3e^4}{4}$

6. $f(x,y) = (x-y)\,e^{-x+y^2}$

$0 \le y \le \sqrt{x}\ ,\ 0 \le x \le 4$



f kont. på slutet begränsat område
$\Rightarrow$ största och minsta värde antas.

I. Inre punkter

$f_1(x,y) = e^{-x+y^2} + (x-y)\cdot e^{-x+y^2}\cdot(-1) = e^{-x+y^2}(1-x+y)$

$f_2(x,y) = -e^{-x+y^2} + (x-y)\cdot e^{-x+y^2}\cdot 2y = e^{-x+y^2}(-1+2xy-2y^2)$

$f_1(x,y)=0: \quad 1-x+y=0 \Rightarrow y=x-1$

$\left.\begin{array}{l} y=x-1 \\ f_2(x,y)=0 \end{array}\right\} \quad -1+2x(x-1)-2(x-1)^2=0 \Rightarrow x=3/2$

Kritisk punkt: $(3/2, 1/2)$

$f(3/2,1/2) = e^{-5/4} \approx 0.29$

II. Randpunkter

A. $y=0\ ,\ 0\le x\le 4$

$f = x\cdot e^{-x} \Rightarrow f' = e^{-x} - xe^{-x} = (1-x)e^{-x}$. $f'=0: 1-x=0 \Rightarrow x=1$

$f(0,0)=0\ ,\ f(1,0)=e^{-1}\approx 0.37\ ,\ f(4,0)=4e^{-4}\approx 0.07$

B. $x=4\ ,\ 0\le y\le 2$

$f=(4-y)e^{-4+y^2} \Rightarrow f' = -e^{-4+y^2} + (4-y)e^{-4+y^2}\cdot 2y = (-1+8y-2y^2)e^{-4+y^2}$

$f'=0: -1+8y-2y^2=0 \Rightarrow y^2-4y+1/2=0 \Rightarrow$

$y = 2\pm\sqrt{4-1/2} \Rightarrow y = 2 \underset{\smile}{\pm} \frac{\sqrt{14}}{2}\ ,\ (0\le y\le 2)$

$f(4,0)=4e^{-4}\approx 0.07\ ,\ f(4,2-\frac{\sqrt{14}}{2}) = (2+\frac{\sqrt{14}}{2})e^{7/2-2\sqrt{14}} \approx 0.07,\ f(4,2)=2$

C. $y=\sqrt{x}\ ,\ 0\le x\le 4$

$f=(x-\sqrt{x})e^{-x+x} = x-\sqrt{x} \Rightarrow f' = 1-\frac{1}{2\sqrt{x}}$

$f'=0: 1-\frac{1}{2\sqrt{x}}=0 \Rightarrow 2\sqrt{x}=1 \Rightarrow x=1/4$

$f(0,0)=0\ ,\ f(1/4,1/2) = -\frac{1}{4}\ ,\ f(4,2)=2$

<u>Svar:</u> Största värde $= 2$ i $(4,2)$

Minsta värde $= -\frac{1}{4}$ i $(1/4, 1/2)$

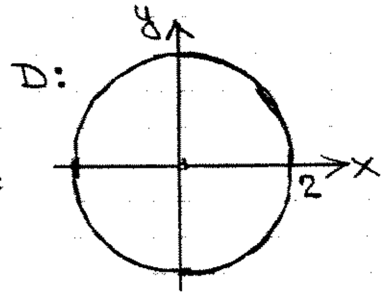
7. a) $z = 16 - x^2 - 2y^2$

$z = 3x^2 + 2y^2$

Skärn. då $3x^2 + 2y^2 = 16 - x^2 - 2y^2$

$\iff \quad x^2 + y^2 = 4$

Volymen $V = \iiint 1 \, dx\,dy\,dz =$



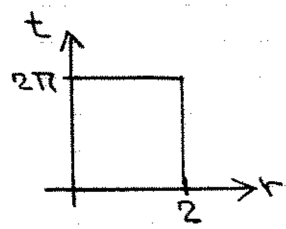
$= \iint_D [z]_{z=3x^2+2y^2}^{z=16-x^2-2y^2} \, dx\,dy =$

$= \iint_D (16 - x^2 - 2y^2 - (3x^2 + 2y^2)) \, dx\,dy =$

$= \iint_D (16 - 4x^2 - 4y^2) \, dx\,dy =$

Låt $\begin{cases} x = r\cos t & ; \; 0 \leq r \leq 2 \\ y = r\sin t & ; \; 0 \leq t \leq 2\pi \end{cases}$

$\frac{\partial(x,y)}{\partial(r,t)} = r$



$= \iint_S (16 - 4r^2\cos^2 t - 4r^2\sin^2 t) \cdot r \, dr\,dt =$

$= \iint_S (16 - 4r^2) \cdot r \, dr\,dt =$

$= \int_0^{2\pi} \left(\int_0^2 (16r - 4r^3) \, dr \right) dt =$

$= \int_0^{2\pi} [8r^2 - r^4]_{r=0}^{r=2} \, dt =$

$= \int_0^{2\pi} (32 - 16) \, dt = \int_0^{2\pi} 16 \, dt =$

$= [16t]_0^{2\pi} = 32\pi$

Svar: Volymen är 32π v.e.

7. b) $z = 16 - x^2 - 2y^2$

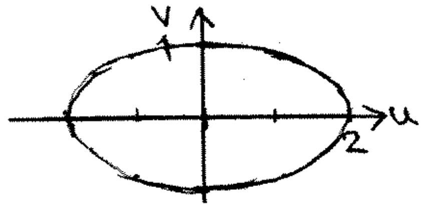
$x^2 + 4y^2 = 4$

Låt $\begin{cases} x = u \\ y = v \\ z = 16 - u^2 - 2v^2 \end{cases}$, $u^2 + 4v^2 = 4$

$\mathbf{r}(u,v) = (u, v, 16 - u^2 - 2v^2)$, $u^2 + 4v^2 = 4$

$$\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2u \\ 0 & 1 & -4v \end{vmatrix} = (2u, 4v, 1)$$

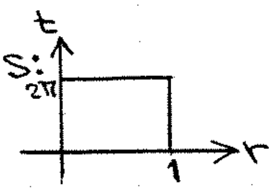
$$\left| \frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv} \right| = \sqrt{4u^2 + 16v^2 + 1}$$



$$\text{Arean} = \iint_D 1 \cdot \sqrt{4u^2 + 16v^2 + 1}\, du\, dv =$$

Låt $\begin{cases} u = 2r\cos t & , \ 0 \le r \le 1 \\ v = r\sin t & , \ 0 \le t \le 2\pi \end{cases}$

$$\frac{\partial(u,v)}{\partial(r,t)} = \begin{vmatrix} 2\cos t & -2r\sin t \\ \sin t & r\cos t \end{vmatrix} = 2r\cos^2 t + 2r\sin^2 t = 2r$$



$$= \iint_S \sqrt{4 \cdot 4r^2\cos^2 t + 16r^2\sin^2 t + 1} \cdot 2r\, dr\, dt =$$

$$= \iint_S \sqrt{16r^2 + 1} \cdot 2r\, dr\, dt =$$

$$= \int_0^1 \left(\int_0^{2\pi} \sqrt{16r^2+1} \cdot 2r\, dt \right) dr =$$

$$= \int_0^1 \left[t \cdot \sqrt{16r^2+1} \cdot 2r \right]_{t=0}^{t=2\pi} dr =$$

$$= 2\pi \int_0^1 \sqrt{16r^2+1} \cdot 2r\, dr =$$

$$= 2\pi \cdot \left[\frac{1}{24} (16r^2+1)^{3/2} \right]_0^1 =$$

$$= \frac{\pi}{12} (17\sqrt{17} - 1)$$

<u>Svar:</u> Arean är $\frac{\pi}{12}(17\sqrt{17} - 1)$

8. $z = x^2 - 2x + y^3 - 1$

Linjen L:s ekvation $\begin{cases} x = 4t \\ y = 3t \\ z = -t \end{cases}$, $t \in \mathbb{R}$

dvs L:s riktningsvektor $u = (4, 3, -1)$.

Tangentplanet är $\perp$ mot linjen L $\Rightarrow$

tangentplanets normalvektor $n = (f_1(a,b), f_2(a,b), -1)$

$\parallel$ linjens riktningsvektor u,

dvs $n = k \cdot u$, $k \in \mathbb{R}$.

$z = x^2 - 2x + y^3 - 1$, $z = f(x,y)$ $\Rightarrow$

$f(x,y) = x^2 - 2x + y^3 - 1$

$f_1(x,y) = 2x - 2 \quad \Rightarrow \quad f_1(a,b) = 2a - 2$

$f_2(x,y) = 3y^2 \quad \Rightarrow \quad f_2(a,b) = 3b^2$

$\because$ $n = k \cdot u$ ger

$(2a-2, 3b^2, -1) = k \cdot (4, 3, -1) \Rightarrow$

$$\begin{cases} 2a - 2 = 4k \\ 3b^2 = 3k \\ -1 = -k \end{cases} \Rightarrow \begin{cases} a = 3 \\ b = \pm 1 \\ k = 1 \end{cases}$$

$\because$ $a = 3$, $b = 1 \Rightarrow f(3,1) = 9 - 6 + 1 - 1 = 3$

$a = 3$, $b = -1 \Rightarrow f(3,-1) = 9 - 6 - 1 - 1 = 1$

Sökta punkter är $(3,1,3)$ och $(3,-1,1)$

<u>Svar:</u> Punkterna är $(3,1,3)$ och $(3,-1,1)$

9. $\int_C \mathbb{F} \cdot d\mathbb{r}$

$\mathbb{F}(x,y,z) = (2xy + z\cos(xz),\ x^2+z,\ y + x\cos(xz))$

$\mathbb{r}(t) = (\cos t, \sin t, t) \quad , \quad 0 \leq t \leq \frac{2\pi}{3}$

$$\begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \mathbb{0} \Rightarrow \frac{\partial}{\partial y}F_3 = \frac{\partial}{\partial z}F_2 ,\ \frac{\partial}{\partial x}F_3 = \frac{\partial}{\partial z}F_1 ,\ \frac{\partial}{\partial x}F_2 = \frac{\partial}{\partial y}F_1$$

$\frac{\partial}{\partial y}F_3 = 1 \qquad \frac{\partial}{\partial x}F_3 = \cos(xz) - xz\sin(xz) \qquad \frac{\partial}{\partial x}F_2 = 2x$

$\frac{\partial}{\partial z}F_2 = 1 \qquad \frac{\partial}{\partial z}F_1 = \cos(xz) - xz\sin(xz) \qquad \frac{\partial}{\partial y}F_1 = 2x$

och enkelt sammanhängande område (hela $\mathbb{R}^3$)

$\Rightarrow$ konservativt vektorfält

$\therefore \quad \int_C \mathbb{F} \cdot d\mathbb{r} = [\phi(x,y,z)]_{P_0}^{P_1}$

Sök nu $\phi(x,y,z)$:

$\frac{\partial \phi}{\partial x} = 2xy + z\cos(xz) \Rightarrow$

$\phi(x,y,z) = \int (2xy + z\cos(xz)\,dx = x^2y + \sin(xz) + C_1(y,z)$

$$\left.\begin{array}{ll} \text{Derivera m.a.p. } y: & \frac{\partial \phi}{\partial y} = x^2 + \frac{\partial C_1(y,z)}{\partial y} \\ \text{Men} & \frac{\partial \phi}{\partial y} = x^2 + z \end{array}\right\} \Rightarrow$$

$\frac{\partial C_1(y,z)}{\partial y} = z \Rightarrow C_1(y,z) = \int z\,dy = yz + C_2(z)$

$\therefore \quad \phi(x,y,z) = x^2y + \sin(xz) + yz + C_2(z)$

$$\left.\begin{array}{ll} \text{Derivera m.a.p } z: & \frac{\partial \phi}{\partial z} = x\cos(xz) + y + C_2'(z) \\ \text{Men} & \frac{\partial \phi}{\partial z} = y + x\cos(xz) \end{array}\right\} \Rightarrow$$

$C_2'(z) = 0 \Rightarrow C_2(z) = \int 0\,dz = C$

$\therefore \quad \phi(x,y,z) = x^2y + \sin(xz) + yz + C$

$\mathbb{r}(t) = (\cos t, \sin t, t) \quad , \quad 0 \leq t \leq \frac{2\pi}{3}$

$t = 0: \quad P_0 = (\cos 0, \sin 0, 0) = (1,0,0)$

$t = \frac{2\pi}{3}: \quad P_1 = (\cos\frac{2\pi}{3}, \sin\frac{2\pi}{3}, \frac{2\pi}{3}) = (-\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{2\pi}{3})$

$\therefore \quad \int_C \mathbb{F} \cdot d\mathbb{r} = [x^2y + \sin(xz) + yz]_{(1,0,0)}^{(-\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{2\pi}{3})} =$

$= \frac{1}{4} \cdot \frac{\sqrt{3}}{2} + \sin(-\frac{\pi}{3}) + \frac{\sqrt{3}}{2} \cdot \frac{2\pi}{3} - (0 + 0 + 0) =$

$= \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{2} + \frac{\pi\sqrt{3}}{3} = \underline{\underline{\frac{\pi\sqrt{3}}{3} - \frac{3\sqrt{3}}{8}}}$