1. Se boken

Lösningsförslag omtenta 27/08/05

9.00–14.00.

2. a) $u(x,y) = xy + x F\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = y + F\left(\frac{y}{x}\right) - \frac{y}{x} F'\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial y} = x + F'\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = xy + x F\left(\frac{y}{x}\right) - \cancel{y F'\left(\frac{y}{x}\right)} + xy + \cancel{y F'\left(\frac{y}{x}\right)}$$

$$= xy + u.$$

b) $\underline{y=0}$ $\lim\limits_{(x,y)\to(0,0)} \frac{\sin\sqrt{|xy|}}{x} = \lim\limits_{x\to 0} \frac{0}{x} = 0.$

$\frac{x=y}{x>0}$ $\lim\limits_{(x,y)\to(0,0)} \frac{\sin\sqrt{|xy|}}{x} = \lim\limits_{\substack{x\to 0\\ x>0}} \frac{\sin|x|}{x} =$

$$= \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

$$\boxed{\left|\frac{\sin xy}{xy}\right| \cdot y \xrightarrow[(x,y)\to 0]{} 0.}$$

3. $f(x,y,z) = x^2 + y^2 + z^2$ finn min

sådan att $5 - xy - z = 0$.

$g(x,y,z) = 5 - xy - z$

$L(x,y,z) = f(x,y,z) + \lambda g(x,y,z)$

Lagrangianen

$$\begin{cases} \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0 \\ \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0 \end{cases}$$

$$\begin{cases} \dfrac{\partial f}{\partial z} + \lambda \dfrac{\partial g}{\partial z} = 0 \\ 5 - xy - z = 0 \end{cases}$$

$$\begin{cases} 2x - \lambda y = 0 \\ 2y - \lambda x = 0 \\ 2z - \lambda = 0 \Rightarrow \boxed{\lambda = 2z} \\ 5 - xy - z = 0 \end{cases}$$

Om $y = 0 \Rightarrow x = 0 \Rightarrow z = 5 \Rightarrow \lambda = 10$.

$(0,0,5,10)$ är stationär punkt till Lagrangianen

$y \neq 0 \Rightarrow \lambda = \dfrac{2x}{y} \Rightarrow 2y - \dfrac{2x}{y} \cdot x = 0$

$\Rightarrow y^2 = x^2 \Rightarrow y = x \Rightarrow \lambda = 2 \Rightarrow z = 1$

$\Rightarrow x^2 = 4 \Rightarrow x = y = 2$; $x = y = -2$

$y = -x$

$\Downarrow$

$\lambda = -2 \Rightarrow z = -1$

$6 + x^2 = 0$ ∴ inga lösningar

$\Rightarrow (0,0,5,10)$; $(2,2,1,2)$ och $(-2,-2,1,2)$ är stationära punkter.

$f(0,0,5) = 25$; $f(2,2,1) = 4+4+1 = 9$

$f(-2,-2,1) = 9$.

4. Ett plan tangerar ytan $x^2+yz+z^2=1$. ~~och~~
Låt (a,b,c) vara koordinaterna av punkten vi söker.

Låt oss betrakta funktionen
$$g(x,y,z)=x^2+yz+z^2-1.$$
$$\nabla g(a,b,c)=(2a,\ c,\ b+2c)$$ och är en normalvektor till ytan $x^2+yz+z^2=1$ i punkten (a,b,c) respektive till tangent planet till ytan i punkten (a,b,c):
$$\nabla g(a,b,c)\perp(a-1,\ b,\ c)$$
och $\nabla g(a,b,c)\perp(a,\ b-1,\ c)$

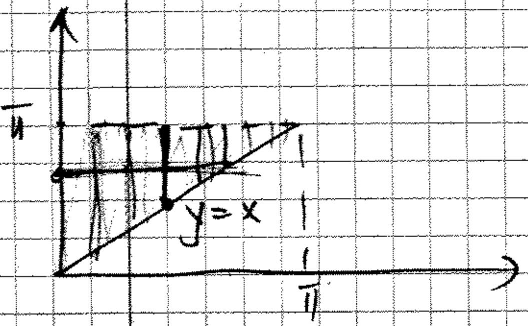
alltså: $$\begin{cases}2a(a-1)+bc+bc+2c^2=0\\2a^2+bc-c+bc+2c^2=0\end{cases}$$

eller punkten (a,b,c) satisfierar de följande tre ekvationerna:
$$\begin{cases}a^2+bc+c^2=1\\2a^2-2a+2bc+2c^2=0 \quad |:2\\2a^2+2bc-c+2c^2=0 \quad |:2\end{cases}$$
$$\begin{cases}a^2+bc+c^2=1\\a^2+bc+c^2-a=0 \quad \Rightarrow a=1\\a^2+bc+c^2-\frac{c}{2}=0 \quad \Rightarrow c=2\end{cases} \Rightarrow b=-2$$

dvs tangeringspunktens koordinater är:

$(1,-2,2)$.

5) Beräkna dubbelintegralen: $\displaystyle\int_0^{\pi}\left(\int_x^{\pi} \frac{\sin^2 y}{y}\,dy\right)dx =$



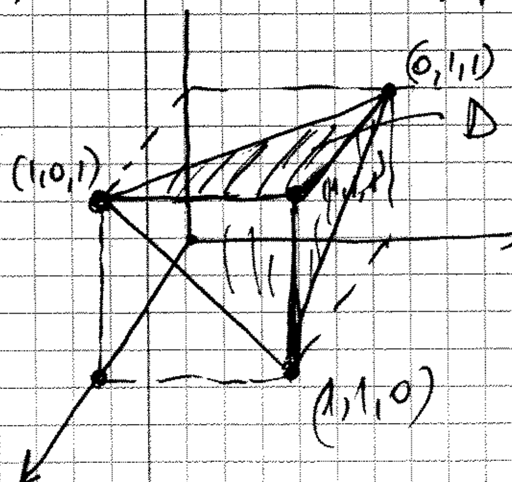
$$= \int_0^{\pi}\left(\int_0^{y} \frac{\sin^2 y}{y}\,dx\right)dy =$$

$$= \int_0^{\pi}\left(\frac{\sin^2 y}{y}\cdot y\right)dy =$$

$$= \int_0^{\pi} \sin^2 y\,dy = \int_0^{\pi} \frac{1-\cos 2y}{2}\,dy$$

$$= \frac{1}{2}y\Big|_0^{\pi} - \frac{\sin 2y}{4}\Big|_0^{\pi} = \frac{\pi}{2}.$$

6) Beräkna trippelintegralen:



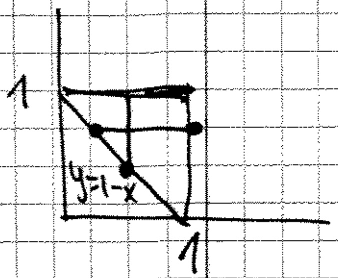
Planet som går genom punkterna $(1,0,1)$, $(0,1,1)$ och $(1,1,0)$ har ekvationen:

$$z+x+y=2.$$

$$\iiint_T y\,dx\,dy\,dz = \iint_D\left(\int_{2-(x+y)}^{1} y\,dz\right)dx\,dy$$

$$= \iint_D y\left(1-2+(x+y)\right)dx\,dy$$

$$= \iint_D y(x+y-1)\,dx\,dy = \int_0^1 \left(\int_{1-y}^{1} y(x+y-1)\,dx \right) dy$$



$$= \int_0^1 \left(y \cdot \frac{x^2}{2}\Big|_{1-y}^{1} + y^2 x \Big|_{1-y}^{1} - yx \Big|_{1-y}^{1} \right) dy$$

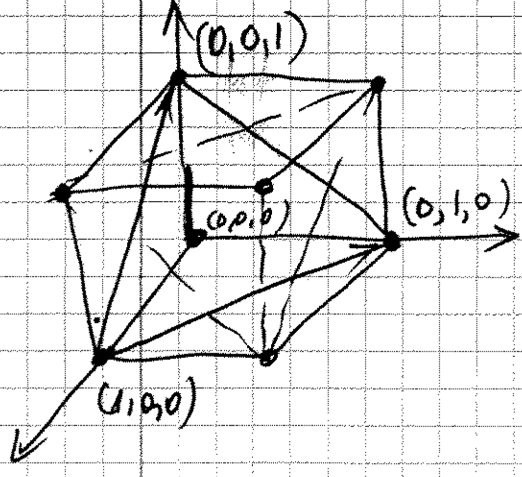
$$= \int_0^1 y\left(\frac{1}{2} - \frac{(1-y)^2}{2}\right) + y^2 - y^2(1-y) - y + y(1-y)$$

$$= \int_0^1 \left(\frac{y}{2} - \frac{y(1-y)^2}{2} + \cancel{y^2} - \cancel{y^2} + y^3 - \cancel{y} + \cancel{y} - y^2 \right) dy$$

$$= \int_0^1 \left[+\frac{y}{2} + y^3 - \frac{y(1+y^2-2y)}{2} - y^2 \right] dy$$

$$= \int_0^1 \left(+\cancel{\frac{y}{2}} + y^3 - \cancel{\frac{y}{2}} - \frac{y^3}{2} + \cancel{y^2} - \cancel{y^2} \right) dy$$

$$= \frac{y^4}{2\cdot 4}\Big|_0^1 = \frac{1}{8} \quad //.$$



två riktningsvektorer

$\bar{u} = (-1,0,1)$ och $\bar{v}(-1,1,0)$

$$\begin{cases} x = 1 + (-1)s - t \\ y = 0 + 0\cdot s + t \\ z = 0 + 1\cdot s + 0t \Rightarrow \boxed{z = s} \end{cases}$$

$\Rightarrow \boxed{y = t} \qquad x = 1 - z - y$

$z = 1 - x - y$

$z = x + y$

$$\iiint_T y\,dx\,dy\,dz = \int_0^1 \int_0^{1-y} \left(\int_0^{1-x-y} y\,dz \right) dx \Big) dy$$

$$= \int_0^1 \left(\int_0^{1-y} y(1-x-y)\,dx \right) dy$$

$$= \int_0^1 y\left(x\Big|_0^{1-y} - \frac{x^2}{2}\Big|_0^{1-y} - yx\Big|_0^{1-y} \right) dy$$

$$= \int_0^1 y\left(1-y-\frac{(1-y)^2}{2} - y(1-y)\right) dy$$

$$= \int_0^1 y\left(1-y-\frac{(1-y)^2}{2} - y + y^2\right) dy$$

$$= \int_0^1 y - 2y^2 + y^3 - \frac{y(1-2y+y^2)}{2}\,dy$$

$$= \int_0^1 \left(y - 2y^2 + y^3 - \frac{y}{2} + y^2 - \frac{y^3}{2}\right) dy$$

$$= \frac{y^4}{2\cdot 4}\Big|_0^1 - \frac{y^3}{3}\Big|_0^1 + \frac{y^2}{2\cdot 2}\Big|_0^1$$

$$= \frac{1}{8} - \frac{1}{3} + \frac{1}{4} = \frac{3-8+6}{24} = \underline{\underline{\frac{1}{24}}}.$$

7)

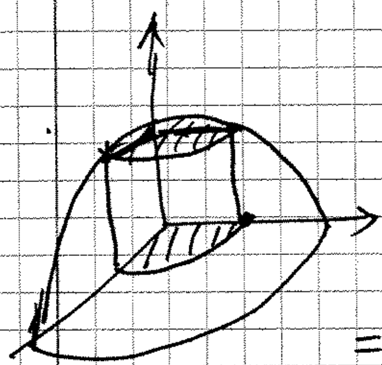
$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} e^{3\cos t + 3\sin t} \cdot (-3\sin t)\cdot 3\cos t +$$

$$\left(d\vec{r} = -3\sin t\,\vec{i} - 3\cos t\,\vec{j} \right)$$

$$+ \; e^{3\cos t + 3\sin t} (-3\sin t)(-3\cos t)\,dt$$

$$= 0.$$

8)



$$\iint_S \frac{1}{\sqrt{x^2+y^2+z^2}}\,dS =$$

$$= \iint_{x^2+y^2\le 1} \frac{1}{\sqrt{x^2+y^2+4-(x^2-y^2)}} \cdot$$

$$\cdot \sqrt{1+\left(\frac{-x}{\sqrt{4-x^2-y^2}}\right)^2 + \left(\frac{-y}{\sqrt{4-x^2-y^2}}\right)^2}\; dx\,dy$$

$$= \iint_{x^2+y^2\le 1} \frac{1}{2}\cdot \sqrt{\frac{4-x^2-y^2+x^2+y^2}{4-x^2-y^2}}\,dx\,dy$$

$$= \iint_{x^2+y^2\le 1} \frac{1}{\sqrt{4-x^2-y^2}}\,dx\,dy = \left(\begin{matrix}\text{polära}\\ \text{koordinater}\end{matrix}\right)$$

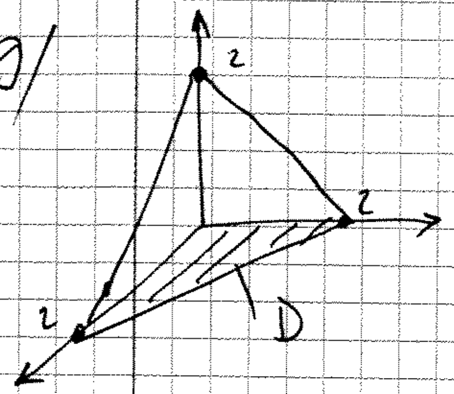
$$= \int_0^1 \int_0^{2\pi} \frac{1}{\sqrt{4-r^2}}\cdot r\,dr\,d\theta = 2\pi\cdot\frac{1}{2}\int_3^4 t^{-\frac{1}{2}}\,dt = \pi\cdot\frac{t^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}$$

$4-r^2=t \quad -2r\,dr=dt$

$r=0\; t=4 \quad r=1\; t=3$

$$= \pi \cdot \frac{\sqrt{t}}{\frac{1}{2}} \Big|_3^4 = 2\pi\left(2-\sqrt{3}\right).$$

9/



$$\iint_S \vec{F} \cdot \vec{N}\, dS = \iint_D (2-x-y, y, x)(1,1,1)\, dx$$

$$= \iint_D (2-x-y+y+x)\, dx\, dy = 2A(D) =$$

$$= 2 \cdot \frac{2 \cdot 2}{2} = 4$$