TENTAMEN i ANALYS B1

för MAB 104, 201, MAB 505

PROBLEM 1.

För $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$,

$$f(x,y) = 1 - \frac{x^2y^3}{x^2+y^2} = 1 - \frac{xy}{x^2+y^2} \cdot xy^2$$

$$g(x,y) := -\frac{xy}{x^2+y^2} \cdot xy^2 .$$

Man vet att $|xy| \le \frac{x^2+y^2}{2}$.

Alltså, $|g(x,y) - 0| = \left|\frac{xy}{x^2+y^2}\right| |xy^2| \le$

$\le \frac{1}{2} |xy^2|$ och $\lim_{(x,y)\to(0,0)} |xy^2| = 0$.

$\Longrightarrow \lim_{(x,y)\to(0,0)} g(x,y) = 0$, som ger

$\lim_{(x,y)\to(0,0)} f(x,y) = 1$. Funktionen

$\tilde{f} : \mathbb{R}^2 \to \mathbb{R}$ är given av:

$$\tilde{f}(x,y) = \begin{cases} \dfrac{x^2+y^2-x^2y^3}{x^2+y^2} & \text{för } (x,y) \neq (0,0) \\ 1 & \text{för } x=y=0. \end{cases}$$

PROBLEM 2.

Man får, successivt:

$$\frac{\partial F}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = 2r\frac{\partial f}{\partial x} + 2s\frac{\partial f}{\partial y}$$

$$(*) \quad \frac{\partial^2 F}{\partial r^2} = 2\frac{\partial f}{\partial x} + 2r\left[\frac{\partial^2 f}{\partial x^2}\cdot 2r + \frac{\partial^2 f}{\partial y \partial x}\cdot 2s\right] +$$

$$+ 2s\left[\frac{\partial^2 f}{\partial x \partial y}\cdot 2r + \frac{\partial^2 f}{\partial y^2}\cdot 2s\right] =$$

$$= 4r^2\frac{\partial^2 f}{\partial x^2} + 4s^2\frac{\partial^2 f}{\partial y^2} + 8rs\frac{\partial^2 f}{\partial x \partial y} + 2\frac{\partial f}{\partial x}.$$

$$\frac{\partial F}{\partial s} = -2s\frac{\partial f}{\partial x} + 2r\frac{\partial f}{\partial y}.$$

$$(**) \quad \frac{\partial^2 F}{\partial s^2} = -2\frac{\partial f}{\partial x} - 2s\left[-2s\frac{\partial^2 f}{\partial x^2} + 2r\frac{\partial^2 f}{\partial y \partial x}\right] +$$

$$+ 2r\left[\frac{\partial^2 f}{\partial x \partial y}\cdot(-2s) + \frac{\partial^2 f}{\partial y^2}\cdot(2r)\right] =$$

$$= -2\frac{\partial f}{\partial x} + 4s^2\frac{\partial^2 f}{\partial x^2} + 4r^2\frac{\partial^2 f}{\partial y^2} - 8rs\frac{\partial^2 f}{\partial y \partial x}$$

Från (*) och (**) fås:

$$\Delta F = 4(r^2+s^2)\frac{\partial^2 f}{\partial x^2} + 4(s^2+r^2)\frac{\partial^2 f}{\partial y^2} =$$

$$= 4(r^2+s^2)\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] = 0$$

därför att f är harmonisk.

Alltså F <u>är också harmonisk</u>.

[Anm: $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ därför att <u>de är kontinuerliga</u>!]

PROBLEM 3.

Det är uppenbar att F har partiella derivator i hela $\mathbb{R}^3$ och de är kontinuerliga. $\boxed{F(4,1,-3)=0}$!

$\frac{\partial F}{\partial y}(x,y,z) = 3xy^2 - 1$ och $\frac{\partial F}{\partial y}(4,1,-3) = 11 \neq 0.$

Enligt implicitafunktionssatsen, $(\exists)$ en omgivning V av punkten $(4,-3)$ (t.ex. en disk med medelpunkt $(4,-3)$ och radie $r>0$), det finns en entydig funktion $y = y(x,z)$ definierad på V med värde i $\mathbb{R}$ så att $y(4,-3) = 1$, $\frac{\partial y}{\partial x}(x,z)$, $\frac{\partial y}{\partial z}(x,z)$ $(\exists)$ i V och

$F(x, y(x,z), z) = 0$ $(\forall)$ $(x,z) \in V$. För $(x,z) \in V$ fås:

$$0 = \frac{\partial 0}{\partial x} = \frac{\partial}{\partial x} F\overbrace{(x, y(x,z), z)}^{A} =$$

$$= \frac{\partial F}{\partial x}(A) + \frac{\partial F}{\partial y}(A)\,\frac{\partial y}{\partial x}(x,z)\left(+\ \frac{\partial z}{\partial x} = 0\right) =$$

$$= \frac{\partial F}{\partial x}(A) + \frac{\partial F}{\partial y}(A)\cdot\frac{\partial y}{\partial x}(x,z).$$

Alltså:

$$\frac{\partial y}{\partial x}(x,z) = -\frac{\frac{\partial F}{\partial x}(A)}{\frac{\partial F}{\partial y}(A)} \quad (\forall)\ (x,z) \in V.$$

Analogt:

$$\frac{\partial y}{\partial z}(x,z) = -\frac{\frac{\partial F}{\partial z}(A)}{\frac{\partial F}{\partial y}(A)}$$

$(\forall)$ $(x,y) \in V$.

PROBLEM 3 (Forts)

$$\begin{cases} \dfrac{\partial y}{\partial x}(x,z) = -\dfrac{[y(x,z)]^3}{3x[y(x,z)]^2-1} \\ \dfrac{\partial y}{\partial z}(x,z) = -\dfrac{1}{3x[y(x,z)]^2+1} \end{cases}, \ (\forall)\ (x,z)\in V.$$

För $(x,z)=(4,-3)$ får vi:

$$\begin{cases} \dfrac{\partial y}{\partial x}(4,-3) = -\dfrac{1}{11} \\ \dfrac{\partial y}{\partial z}(4,-3) = -\dfrac{1}{11} \end{cases}$$

Tangentplanet till grafen av y i $(4,1,-3)$ har ekvationen:

$$y-1 = \frac{\partial y}{\partial x}(4,-3)(x-4) + \frac{\partial y}{\partial z}(4,-3)(z+3) \Leftrightarrow$$

$$y-1 = -\frac{1}{11}(x-4) - \frac{1}{11}(z+3) \Leftrightarrow$$

$$\boxed{x+z+11y-12=0}.$$

PROBLEM 4.

Kurvan $C = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+2y^2+2z^2=8;\ z=x+y\}$

är en sluten och begränsad mängd.

Funktionen f är kontinuerlig.

Enligt Weierstrass's sats, det finns två punkter $(x_1,y_1,z_1) = P_1$, $(x_2,y_2,z_2) = P_2 \in C$ så att

$$f(P_1) = \min_{P\in C} f(P) \text{ och } f(P_2) = \max_{P\in C} f(P).$$

Punkterna P_1 och P_2 är <u>kritiska punkter</u> till Lagrange's funktion

$$L(x,y,z,\lambda,\mu) = f(x,y,z) + \lambda(x^2+2y^2+2z^2-8) + \mu(x+y-z).$$

$$\frac{\partial L}{\partial x} = 0;\ \frac{\partial L}{\partial y} = 0;\ \frac{\partial L}{\partial z} = 0;\ \frac{\partial L}{\partial \lambda} = 0;\ \frac{\partial L}{\partial \mu} = 0. \Leftrightarrow$$

$$\begin{cases} 1 + 2\lambda x + \mu = 0 \\ 4\lambda y + \mu = 0 \\ 4\lambda z - \mu = 0 \\ x^2+2y^2+2z^2-8 = 0 \\ x+y-z = 0 \end{cases} \Longrightarrow \begin{cases} P_1 = (x_1,y_1,z_1) = (-2,+1,-1) \\ P_2 = -P_1 = (2,-1,1) \end{cases}$$

$$\boxed{f(P_1) = -2\ ;\ f(P_2) = 2} \Leftrightarrow$$

$$\boxed{\max_{(x,y,z)\in C} x = 2;\ \min_{(x,y,z)\in C} x = -2}$$

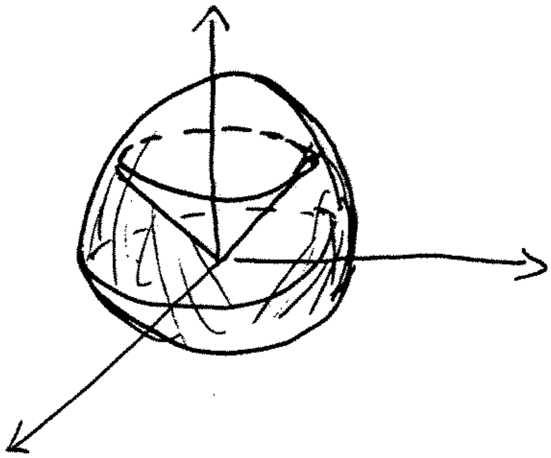
5. Evaluate

$$I = \iiint_G (x+z)\, dx\, dy\, dz \quad ,$$

where G is the region defined by

$$x^2 + y^2 + z^2 \le R^2 \ , \quad z \le \sqrt{x^2+y^2} \ .$$

Solution. The region G consists of all points of the sphere $x^2+y^2+z^2 \le R^2$ that lie outsid
of the interior of the cone $z > \sqrt{x^2+y^2}$.

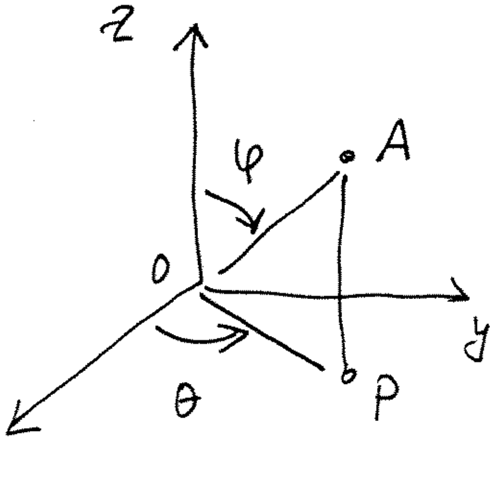


We use the spherical coordinates

$$x = \rho \cos\theta \sin\varphi$$

$$y = \rho \sin\theta \sin\varphi$$

$$z = \rho \cos\varphi$$



In spherical coordinats G is described by

$$\frac{\pi}{4} \le \varphi \le \pi \ , \quad 0 \le \theta \le 2\pi \quad , \quad 0 \le \rho \le R \ .$$

Observe that by symmetry

$$\iiint\limits_G x\,dx\,dy\,dz = 0 .$$

On the other hand, in spherical coordinates

$$\iiint\limits_G z\,dx\,dy\,dz = \int_0^{2\pi} d\theta \int_{\pi/4}^{\pi} \cos\varphi \sin\varphi\, d\varphi \int_0^R \rho^3 d\rho =$$

$$= \frac{\pi R^2}{4} \int_{\pi/4}^{\pi} \sin 2\varphi\, d\varphi = \frac{\pi R^4}{8} .$$

Answer : $I = \dfrac{\pi R^4}{8}$.

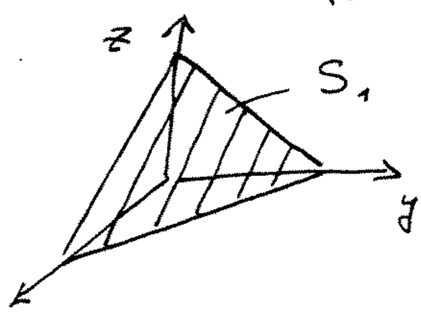
6 . Evaluate the surface integral

$$\iint\limits_S (x+y+2z)\,d\sigma ,$$

where S is the total surface of the tetrahedron

$$x+y+z \le 1 , \quad x \ge 0,\ y \ge 0,\ z \ge 0 .$$

Solution . The surface S consists of 4 pieces (4 triangles) .

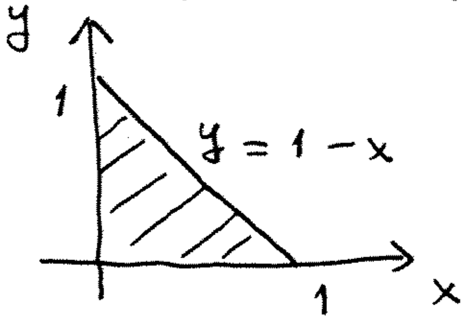


1) S_1 : $x+y+z = 1$,
$x \ge 0 , y \ge 0 , z \ge 0$.

We have $z = 1-x-y$, $z'_x = z'_y = -1$,

$$d\sigma = \sqrt{3}\, dx\, dy,$$

$$f(x,y,z) = x+y+2z = 1-x-y \quad \text{on } S_1 .$$



Thus,

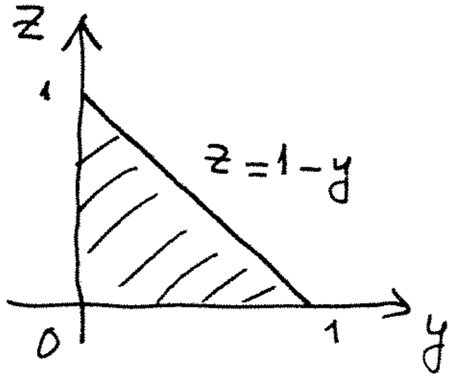
$$\iint_{S_1} (x+y+2z)\, d\sigma =$$

$$= \sqrt{3} \int_0^1 dx \int_0^{1-x} (1-x-y)\, dy = \frac{\sqrt{3}}{6} .$$

2) S_2 (on the plane Oyz) :

$$x = 0, \quad y+z \le 1, \quad y \ge 0, \quad z \ge 0 .$$

We have $d\sigma = dy\, dz$ and

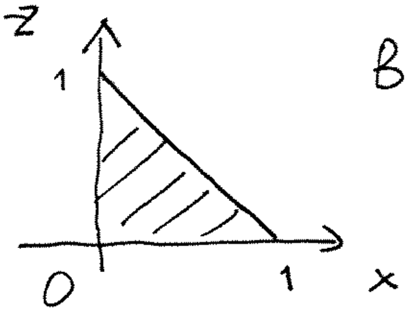


$$\iint_{S_2} (x+y+2z)\, d\sigma =$$

$$= \int_0^1 dy \int_0^{1-y} (y+2z)\, dz =$$

$$= \int_0^1 \left[(1-y)y + (1-y)^2 \right] dy = \frac{1}{2}$$

3) S_3 (on the plane Oxz)



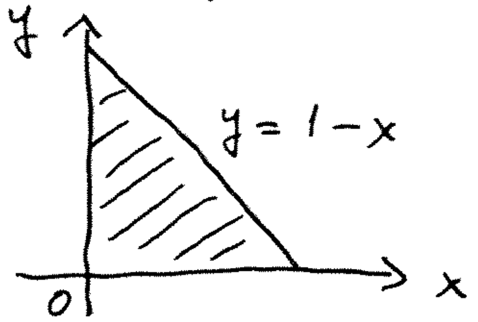
By symmetry,

$$\iint_{S_3} (x+y+2z)\, d\sigma = \iint_{S_2} (x+y+2z)\, d\sigma$$

$$= \frac{1}{2}$$

4) S_4 (on the plane Oxy):

$$z=0, \quad x+y\le 1, \quad x\ge 0, \; y\ge 0$$



We have

$$\iint_{S_4}(x+y+2z)\,d\sigma = \int_0^1 dx \int_0^{1-x}(x+y)\,dy =$$

$$= \int_0^1 \left[x(1-x) + \frac{(1-x)^2}{2} \right] dx = \frac{1}{3}.$$

Answer: $\dfrac{8+\sqrt{3}}{6}$.

7) Evaluate the integral

$$I = \iint_D \frac{1}{x^2y^2}\,dx\,dy,$$

where D is the region bounded by straight lines

$$3y = x, \; y = 3x, \; y = 4-5x, \; y = 4-x.$$

Solution. Introduce parameters u and v:

$$y = ux, \quad y = 4 - vx.$$

We have:

$$ux = 4 - vx .$$

From here

$$\begin{cases} x = \dfrac{4}{u+v} \\ y = \dfrac{4u}{u+v} \end{cases}$$

and we have that in new coordinates

$$\frac{1}{3} \leq u \leq 3 \ , \quad 1 \leq v \leq 5 .$$

Next, the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = 16 \begin{vmatrix} -\dfrac{1}{(u+v)^2} & -\dfrac{1}{(u+v)^2} \\ \dfrac{v}{(u+v)^2} & -\dfrac{u}{(u+v)^2} \end{vmatrix} =$$

$$= \frac{16}{(u+v)^3} .$$

Subintegral function

$$\frac{1}{x^2 y^2} = \frac{(u+v)^4}{16^2 u^2} .$$

Thus, we get

$$I = \frac{1}{16} \int_{1/3}^{3} du \int_{1}^{5} \frac{u+v}{u^2}\, dv = \frac{4+\ln 3}{2}.$$

Answer: $\dfrac{4+\ln 3}{2}$

8. Evaluate

$$\int_C z^2 dx + x dy$$

where C is the curve of intersection of hemisphere

$$x^2+y^2+z^2=4, \quad z \geq 0,$$

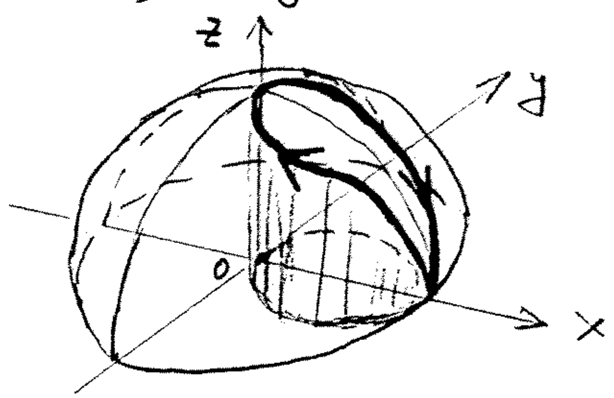
and cylinder $x^2+y^2=2x$, running counter-clockwise, if look from the origin.

Solution. The directrix of the cylinder is the circle

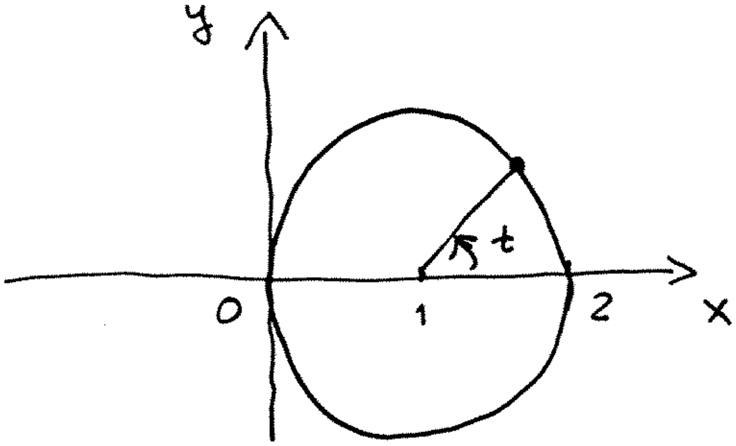
$$x^2+y^2=2x$$

or, equivalently,

$$(x-1)^2+y^2=1.$$



The direction on the curve is indicated by arrows. We introduce a parametrization of the curve.



We have
$$x = 1 + \cos t, \quad y = \sin t.$$

Next,
$$z = \sqrt{4 - x^2 - y^2} = 2 \sin \frac{t}{2}.$$

Thus, the parametric equations of the curve can be chosen as

$$x = 1 + \cos t, \quad y = \sin t, \quad z = 2 \sin \frac{t}{2}; \quad 0 \le t \le 2\pi.$$

But this parametrization determines the opposite orientation of the curve. We will take this into account, changing the sign. We have

$$\int_C z^2 dx + x dy = - \int_0^{2\pi} [2(1-\cos t)(-\sin t) + (1+\cos t)\cos t]$$

$$= 2 \int_0^{2\pi} (1-\cos t) \sin t \, dt - \int_0^{2\pi} \cos t \, dt - \int_0^{2\pi} \cos^2 t \, dt = -\pi$$

<u>Answer</u> : $I = -\pi$.