Analys B1 Lösningsförslag Tentamen 060327

1 a) Låt C_1 och C_2 vara två (styckvis glatta) kurvor i D, som startar i punkten P_0 och slutar i punkten P_1. Då är kurvan $C = C_1 - C_2$ sluten (och styckvis glatt), dvs. $\oint_C \mathbb{F} \cdot d\mathbf{r} = 0$.

Men $\oint_C \mathbb{F} \cdot d\mathbf{r} = \int_{C_1} \mathbb{F} \cdot d\mathbf{r} + \int_{-C_2} \mathbb{F} \cdot d\mathbf{r} =$

$= \int_{C_1} \mathbb{F} \cdot d\mathbf{r} - \int_{C_2} \mathbb{F} \cdot d\mathbf{r}$.

$\therefore \int_{C_1} \mathbb{F} \cdot d\mathbf{r} = \int_{C_2} \mathbb{F} \cdot d\mathbf{r}$

b) $\mathbf{r}(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 2\pi$.

$\mathbf{r}'(t) = (-\sin t, \cos t, 1)$

$|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$\therefore$ Längden $= \int_0^{2\pi} \sqrt{2}\, dt = \underline{\underline{2\sqrt{2}\pi}}$.

2) $f(x,y) = \dfrac{x^2 + 2y^2}{3x^2 + y^2}$

a) Längs linjen $x = 0$: $f(0,y) = \dfrac{2y^2}{y^2} = 2 \to 2$ när $y \to 0$

Längs linjen $y = 0$: $f(x,0) = \dfrac{x^2}{3x^2} = \dfrac{1}{3} \to \dfrac{1}{3}$ när $x \to 0$

$\dfrac{1}{3} \neq 2 \Rightarrow \lim_{(x,y)\to(0,0)} f(x)$ existerar ej.

b) $f'_x = \dfrac{2x(3x^2 + y^2) - 6x(x^2 + 2y^2)}{(3x^2 + y^2)^2} = \dfrac{-10xy^2}{(3x^2+y^2)^2}$

$\Rightarrow f'_x(-1,1) = \dfrac{10}{16} = \underline{\underline{\dfrac{5}{8}}}$

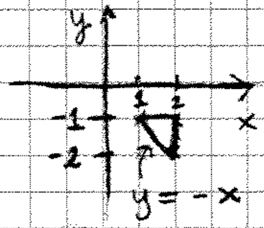
$f'_y = \dfrac{4y(3x^2 + y^2) - 2y(x^2 + 2y^2)}{(3x^2 + y^2)^2} = \dfrac{10yx^2}{(3x^2+y^2)^2}$

$\Rightarrow f'_y(-1,1) = \dfrac{10}{16} = \underline{\underline{\dfrac{5}{8}}}$

c) Låt $v = (3,-4)$. $|v| = \sqrt{3^2+4^2} = \sqrt{25} = 5$

$$\therefore \quad \frac{df}{ds}(-1,1) = \nabla f(-1,1)\cdot \frac{v}{|v|} = \left(\frac{5}{8},\frac{5}{8}\right)\cdot \frac{(3,-4)}{5} = \underline{\underline{-\frac{1}{8}}}$$

3)



$$\iint_E \frac{1}{1+x+y}\,dx\,dy = \int_1^2\int_{-x}^{-1} \frac{1}{1+x+y}\,dy\,dx =$$

$$= \int_1^2 \Big[\ln|1+x+y|\Big]_{-x}^{-1} dx = \int_1^2 \ln x - \underbrace{\ln 1}_{=0}\,dx =$$

$$= \int_1^2 1\cdot \ln x\,dx = \Big[x\ln x\Big]_1^2 - \int_1^2 x\cdot\frac{1}{x}\,dx = 2\ln 2 - 1\ln 1 - (2-1) =$$

$$= \underline{\underline{2\ln 2 - 1}}.$$

4)

$$f'_y = f'_u\cdot u'_y + f'_v\cdot v'_y = f'_v$$

$$f''_{yy} = f''_{vu}\,u'_y + f''_{vv}\,v'_y = f''_{vv}$$

$$f''_{yx} = f''_{vu}\,u'_x + f''_{vv}\,v'_x = \frac{1}{x}f''_{vu} - \frac{1}{x}f''_{vv}$$

$$\therefore \quad x f''_{xy} + f''_{yy} = f''_{uv} - f''_{vv} + f''_{vv} = f''_{uv}$$

$\boxed{\text{Alt}}$

$$\begin{cases} u = \ln x \\ v = y - \ln x \end{cases} \iff \begin{cases} x = e^u \\ y = u+v \end{cases}$$

$$f'_u = f'_x\cdot x'_u + f'_y\cdot y'_u = f'_x\cdot e^u + f'_y$$

$$f''_{uv} = \left(f''_{xx}\cdot x'_v + f''_{xy}\cdot y'_v\right)\cdot e^u + f''_{yx}\cdot x'_v + f''_{yy}\cdot y'_v =$$

$$= f''_{xy}\,e^u + f''_{yy} = x f''_{xy} + f''_{yy}$$

5) Sätt $F(x,y,z) = xz^3 - x^3y^2 + yz$

$$F'_z = 3xz^2 + y \implies F'_z(1,-2,2) = 10 \neq 0$$

$\therefore$ Implicita funktionssatsen ger att ekvationen definierar z som en funktion av x och y, $z = f(x,y)$, nära punkten $(1,-2,2)$

$$F'_y = -2x^3y + z \implies F'_y(1,-2,2) = 4+2 = 6$$

$$\therefore \quad \frac{\partial f}{\partial y}(1,-2) = -\frac{F'_y(1,-2,2)}{F'_z(1,-2,2)} = -\frac{6}{10} = \underline{\underline{-\frac{3}{5}}}$$

$\boxed{\text{Alt}}$ Derivera $xz^3 - x^3y^2 + yz = 0$ implicit m.a.p. y:

$$3xz^2\cdot z'_y - 2x^3y + z + yz'_y = 0$$

$$(x,y,z) = (1,-2,2) \implies 12z'_y + 4 + 2 - 2z'_y = 0 \iff z'_y = -\frac{6}{10} = \underline{\underline{-\frac{3}{5}}}$$

6) Undersök om $\mathbb{F}$ har en skalärpotential ϕ, dvs. om det finns en funktion $\phi = \phi(x,y,z)$ s.a $\nabla \phi = \mathbb{F}$.

$$\begin{cases} \dfrac{\partial \phi}{\partial x} = 3x^2 z e^{x^3} & (1) \\ \dfrac{\partial \phi}{\partial y} = y & (2) \\ \dfrac{\partial \phi}{\partial z} = e^{x^3} & (3) \end{cases}$$

Integrera (1) m.a.p. x: $\phi = z e^{x^3} + h(y,z)$

Sätt in i (2): $\dfrac{\partial \phi}{\partial y} = h'_y = y \Leftrightarrow h = \dfrac{y^2}{2} + g(z)$

$\therefore \phi = z e^{x^3} + \dfrac{y^2}{2} + g(z)$

Sätt in i (3):

$\dfrac{\partial \phi}{\partial z} = z e^{x^3} + g' = e^{x^3} \Leftrightarrow g' = 0 \Leftrightarrow g = C$

Sätt tex. $C = 0$.

$\therefore \phi = z e^{x^3} + \dfrac{y^2}{2} \; (+C)$ och $\mathbb{F}$ är konservativt.

Startpunkt $(t=0)$: $(0,0,0)$

Slutpunkt $(t=1)$: $(1,1,1)$

$\therefore \displaystyle\int_C \mathbb{F} \cdot d\mathbf{r} = \phi(1,1,1) - \phi(0,0,0) = \underline{\underline{e + \frac{1}{2}}}$

$\boxed{\text{Alt.}}$ $\mathbf{r}(t) = (t, t^2, t^3) \Rightarrow \mathbf{r}'(t) = (1, 2t, 3t^2)$

$$\therefore \int_C \mathbb{F} \cdot d\mathbf{r} = \int_0^1 \mathbb{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt =$$

$$= \int_0^1 (3t^5 e^{t^3}, t^2, e^{t^3}) \cdot (1, 2t, 3t^2)\, dt =$$

$$= \int_0^1 3t^5 e^{t^3} + 2t^3 + 3t^2 e^{t^3}\, dt = \int_0^1 3t^5 e^{t^3}\, dt +$$

$$+ \left[\frac{2t^4}{4}\right]_0^1 + \left[e^{t^3}\right]_0^1 = \left[\begin{array}{l} s = t^3 \\ ds = 3t^2 dt \\ t=0 \Rightarrow s=0 \\ t=1 \Rightarrow s=1 \end{array}\right] = \int_0^1 s e^s\, ds +$$

$$+ \frac{1}{2} + (e - e^0) + \left[se^s\right]_0^1 - \int_0^1 e^s\, ds = e - \frac{1}{2} + e - \left[e^s\right]_0^1 =$$

$$= e - \frac{1}{2} + e - (e-1) = \underline{\underline{e + \frac{1}{2}}}$$

7a) $x^2+y^2+z^2=1 \vee z=\sqrt{x^2+y^2} \Longrightarrow$

$\Longrightarrow 2z^2=1 \Longleftrightarrow z=\cancel{\pm}\frac{1}{\sqrt{2}}$ ty $z\geq 0$.

Sfäriska koordinater: $\begin{cases} x=r\sin\phi\cos\theta \\ y=r\sin\phi\sin\theta \\ z=r\cos\phi \end{cases}$, $0\leq r\leq 1$, $0\leq\phi\leq\frac{\pi}{4}$, $0\leq\theta\leq 2\pi$

$$V=\iiint_{\substack{x^2+y^2+z^2\leq 1,\\ z\geq\sqrt{x^2+y^2}}} dV = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 r^2\sin\phi\, dr\, d\phi\, d\theta = 2\pi\left[\frac{r^3}{3}\right]_0^1\left[-\cos\phi\right]_0^{\pi/4} =$$

$$= 2\pi\cdot\frac{1}{3}\left(1-\frac{1}{\sqrt{2}}\right)=\underline{\underline{\frac{\pi}{3}(2-\sqrt{2})}}$$

$\boxed{A}$ $V=\iiint_{\substack{x^2+y^2+z^2\leq 1\\ z\geq\sqrt{x^2+y^2}}} dV = \Big[$ Cylindriska koordinater: $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$, $0\leq r\leq 1/\sqrt{2}$, $0\leq\theta\leq 2\pi$, $r\leq z\leq\sqrt{1-r^2}$ $\Big] =$

$$=\int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_r^{\sqrt{1-r^2}} r\, dz\, dr\, d\theta = 2\pi\int_0^{1/\sqrt{2}} r\sqrt{1-r^2}-r^2\, dr =$$

$$=2\pi\left[-\frac{(1-r^2)^{3/2}}{3}-\frac{r^3}{3}\right]_0^{1/\sqrt{2}} = 2\pi\left(-\frac{1}{3\cdot 2\sqrt{2}}-\frac{1}{3\cdot 2\sqrt{2}}+\frac{1}{3}\right)=$$

$$=\underline{\frac{\pi}{3}(2-\sqrt{2})}$$

b) Sätt $G(x,y,z)=x^2+y^2+z^2$

$\nabla G(x,y,z)=(2x,2y,2z)$

$\therefore$ En normal till ytan är $n=(x,y,z)$

$$\left|\frac{n}{n_3}\right|=\frac{\sqrt{x^2+y^2+z^2}}{z}=\frac{\sqrt{1}}{\sqrt{1-(x^2+y^2)}}$$

ty $z\geq 0$

$$\therefore \text{Arean}=\iint_S dS=\iint_{\sqrt{x^2+y^2}\leq\frac{1}{\sqrt{2}}}\left|\frac{n}{n_3}\right| dx\, dy=\iint_{\sqrt{x^2+y^2}\leq\frac{1}{\sqrt{2}}}\frac{1}{\sqrt{1-(x^2+y^2)}}\, dx\, dy=$$

$$=\int_0^{2\pi}\int_0^{1/\sqrt{2}}\frac{1}{\sqrt{1-r^2}}\, r\, dr\, d\theta = 2\pi\left[-\sqrt{1-r^2}\right]_0^{1/\sqrt{2}}=$$

$$=2\pi\left(-\frac{1}{\sqrt{2}}+1\right)=\underline{\underline{\pi(2-\sqrt{2})}}$$

7b) $\boxed{\text{Alt.}}$ $\mathbf{r}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$, $0 \le u \le \pi/4$,
$0 \le v \le 2\pi$.

$\mathbf{r}'_u = (\cos u \cos v, \cos u \sin v, -\sin u)$

$\mathbf{r}'_v = (-\sin u \sin v, \sin u \cos v, 0)$

$$\mathbf{r}'_u \times \mathbf{r}'_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos u \cos v & \cos u \sin v & -\sin u \\ -\sin u \sin v & \sin u \cos v & 0 \end{vmatrix} =$$

$$= (\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u (\cos^2 v + \sin^2 v)) =$$

$$= (\sin^2 u \cos v, \sin^2 u \sin v, \cos u \sin u)$$

$$|\mathbf{r}'_u \times \mathbf{r}'_v|^2 = \sin^4 u \cos^2 v + \sin^4 u \sin^2 v + \cos^2 u \sin^2 u =$$

$$= \sin^2 u (\sin^2 u + \cos^2 u) = \sin^2 u$$

$\therefore$ $|\mathbf{r}'_u \times \mathbf{r}'_v| = \sin u$

$$\text{Arean} = \int_0^{2\pi}\int_0^{\pi/4} \sin u \, du \, dv = 2\pi \left[-\cos u\right]_0^{\pi/4} = 2\pi\left(-\frac{1}{\sqrt{2}} + 1\right) =$$
$$= \underline{\underline{\pi(2-\sqrt{2})}}$$

8 a) $\begin{cases} z'_x = y \\ z'_y = x \end{cases}$ Punkten (a,b,c) ligger på ytan $z = xy \Rightarrow$
$\Rightarrow c = ab$

$\therefore$ Tangentplanets ekvation är $z = c + b(x-a) + a(y-b)$
$\underset{c=ab}{\Leftrightarrow}$ $\underline{\underline{z = bx + ay - ab}}$

$\boxed{\text{Alt.}}$ Sätt $G(x,y,z) = xy - z$

$\nabla G = (y, x, -1)$ är en normalvektor till funktionsytan $\Rightarrow$ $\nabla G(a,b,c) = (b, a, -1)$ är en normalvektor till tangentplanet.

$\therefore$ Tangentplanets ekvation: $bx + ay - z + D = 0$

Sätt in punkten $(x,y,z) = (a,b,c)$:

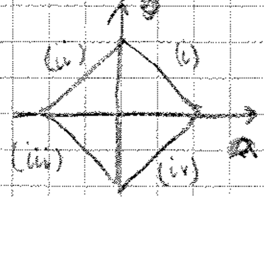
$ba + ab - c + D = 0 \Leftrightarrow D = c - 2ab$

$\therefore$ Tangentplanets ekvation är $bx + ay - z + c - 2ab =$

$\underset{c=ab}{\Leftrightarrow}$ $\underline{\underline{z = bx + ay - ab}}$

b) $(x,y,z) = (0,0,w) \Rightarrow w = -ab$.

Området slutet och begränsat (kompakt)
$\Rightarrow \exists$ största och minsta värde!



1) Stationära punkter: $\begin{cases} w'_a = b \\ w'_b = a \end{cases}$ $\therefore$ $\begin{cases} w'_a = 0 \\ w'_b = 0 \end{cases}$ $\Leftrightarrow$ $(a,b) = (0,0)$

2) Singulär punkter saknas.

3) Randen: Pg.a symmetri är w lika stor på randdelen (iii) som på (i) och lika stor på (iv) som på (ii). Vi kan också utnyttja att w är lika stor, men med ombytt tecken, på (ii) som på (i).

(i) $b = 2 - a$, $0 \le a \le 2$.

$w(a, 2-a) = 2a - a^2$

$w' = 2 - 2a \Leftrightarrow a = 1.$

<u>Kandidater:</u> $(1,1)$, $(0,2)$, $(2,0)$

$$\left.\begin{array}{ll} f(0,0) = 0 & \\ f(1,1) = -1 & (f(-1,1) = 1) \\ f(0,2) = 0 & (f(0,2) = 0) \\ f(2,0) = 0 & (f(-2,0) = 0) \end{array}\right\} \text{från randdel (ii)}$$

$\therefore$ Minsta värdet är -1 och största värdet är 1.

7) $\boxed{\text{Alt 2}}$ $\mathbf{r}(u,v) = (u, v, \sqrt{1-u^2-v^2})$, (ty $z \geq 0$)

$$\mathbf{r}'_u = \left(1, 0, \frac{-u}{\sqrt{1-u^2-v^2}}\right)$$

$$\mathbf{r}'_v = \left(0, 1, \frac{-v}{\sqrt{1-u^2-v^2}}\right)$$

$$\mathbf{r}'_u \times \mathbf{r}'_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -u/\sqrt{1-u^2-v^2} \\ 0 & 1 & -v/\sqrt{1-u^2-v^2} \end{vmatrix} = \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1\right)$$

$$|\mathbf{r}'_u \times \mathbf{r}'_v| = \sqrt{\frac{u^2}{1-u^2-v^2} + \frac{v^2}{1-u^2-v^2} + 1} = \frac{1}{\sqrt{1-u^2-v^2}}$$

$$\therefore \text{Arean} = \iint_{u^2+v^2 \le \frac{1}{2}} \frac{1}{\sqrt{1-u^2-v^2}}\,du\,dv = \int_0^{2\pi}\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1-r^2}}\,r\,dr\,d\theta = 2\pi\left[-\sqrt{1-r^2}\right]_0^{1/\sqrt{2}} =$$

$$= \underline{\underline{\pi(2-\sqrt{2})}}$$