Lösningsförslag Analys B1 Tentamen 060429

1) Se kurslitteraturen.

2) a) Längs linjen $y = 1$: $f(x,1) = \dfrac{x-1}{x-1} = 1 \to 1$ då $x \to 1$.

Längs linjen $y = x$: $f(x,x) = \dfrac{0}{x-1} = 0 \to 0 \neq 1$ då $x \to 1$

$\therefore$ Gränsvärdet saknas då $(x,y) \to (1,1)$.

b) $v = (2,1) \Rightarrow |v| = \sqrt{2^2+1^2} = \sqrt{5}$

$$f'_x = \frac{(x-1)-(x-y)}{(x-1)^2} = \frac{y-1}{(x-1)^2} \Rightarrow f'_x(2,-1) = -2$$

$$f'_y = \frac{-1}{x-1} \Rightarrow f'_y(2,-1) = -1$$

$$\therefore \frac{df}{ds}(2,-1) = \nabla f(2,-1) \cdot \frac{v}{|v|} = (-2,-1)\cdot\frac{(2,1)}{\sqrt{5}} = -\frac{5}{\sqrt{5}} = \underline{\underline{-\sqrt{5}}}$$

c) $\dfrac{df}{ds} = \nabla f \cdot \dfrac{v}{|v|} = |\nabla f| \cos\theta$ där θ är vinkeln mellan riktningen v och ∇f så $0 \leq \theta \leq \pi$

$\dfrac{df}{ds}$ är minst när $\cos\theta = -1$, dvs när $\theta = \pi$.

$\therefore$ Funktionen avtar snabbast i motsatt riktning mot $\nabla f(2,-1)$, dvs i riktningen $\underline{\underline{(2,1)}}$.

d) $$f''_{xx} = -2\,\frac{y-1}{(x-1)^3} \Rightarrow f''_{xx}(2,-1) = 4$$

$$f''_{xy} = \frac{1}{(x-1)^2} \Rightarrow f''_{xy}(2,-1) = 1$$

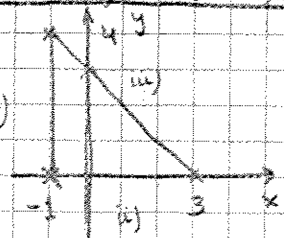
$$f''_{yy} = 0 \Rightarrow f''_{yy}(2,-1) = 0$$

$f(2,-1) = 3$, $f'_x(2,-1) = 2$, $f'_y(2,-1) = -1$

$$\therefore P_2(x,y) = 3 - 2(x-2) - (y+1) + \frac{4(x-2)^2 + 2\cdot(x-2)(y+1)}{2!}$$

$$= \underline{3 - 2(x-2) - (y+1) + 2(x-2)^2 + (x-2)(y+1)}$$

3) $f(x,y) = x^2 + 2xy + 4y^2 + 6y$



f kontinuerlig på slutet och begränsat (kompakt) område, så största och minsta värde existerar.

(1) Stationära punkter:

$$\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases} \Leftrightarrow \begin{cases} 2x + 2y = 0 \\ 2x + 8y + 6 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -y \\ 6y = -6 \end{cases} \Leftrightarrow \begin{cases} x = 1 \\ y = -1 \end{cases}$$

$(1,-1)$ tillhör ej området.

(2) Singulära punkter saknas

(3) Randen:

i) $x = -1$: $g(y) = f(-1,y) = 1 + 4y + 4y^2$, $0 \le y \le 4$

$g' = 0 \Leftrightarrow 4 + 8y = 0 \Leftrightarrow y = -\frac{1}{2} < 0$

Kandidater: $(-1,0)$, $(-1,4)$

ii) $y = 0$: $h(x) = f(x,0) = x^2$, $-1 \le x \le 3$

$h' = 0 \Leftrightarrow 2x = 0 \Leftrightarrow x = 0$

Kandidater: $(0,0)$, $(-1,0)$, $(3,0)$

iii) $y = 3 - x$: $\varphi(x) = f(x, 3-x) = x^2 + 2x(3-x) + 4(3-x)^2 + 6(3-x) =$

$= x^2 + 6x - 2x^2 + 4(9 - 6x + x^2) + 18 - 6x = 3x^2 - 24x + 54$, $-1 \le x \le 3$

$\varphi' = 0 \Leftrightarrow 6x - 24 = 0 \Leftrightarrow x = 4 > 3$

Kandidater: $(3,0)$, $(-1,4)$

$f(-1,0) = 1$

$f(-1,4) = 1 + 16 + 64 = 81$

$f(0,0) = 0$

$f(3,0) = 9$

Svar: Största värdet är $\underline{\underline{81}}$ och minsta värdet är $\underline{\underline{0}}$.

4) $\iiint_T \frac{\sin(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}}\,dx\,dy\,dz = \left[\text{Sfäriska koordinater} \right] =$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} \frac{\sin r^2}{r}\, r^2 \sin\phi \, dr\, d\phi\, d\theta = \frac{\pi}{2}\int_0^{\pi/2} \sin\phi\, d\phi \int_{\sqrt{\pi/2}}^{\sqrt{\pi}} r \sin r^2\, dr =$$

$$= \frac{\pi}{2}\left[-\cos\phi\right]_0^{\pi/2} \left[-\frac{\cos r^2}{2}\right]_{\sqrt{\pi/2}}^{\sqrt{\pi}} = \frac{\pi}{2}\cdot 1 \cdot \frac{1}{2} = \underline{\underline{\frac{\pi}{4}}}$$

5) $g'_t = g'_u \cdot u'_t + g'_v \cdot v'_t = g'_v$

$g''_{tt} = g''_{vu}\, u'_t + g''_{vv}\, v'_t = g''_{vv}$

$g''_{tx} = g''_{vu} \cdot u'_x + g''_{vv} \cdot v'_x = g''_{vu} - 2x\, g''_{vv} = g''_{vu} - 2u\, g''_{vv}$

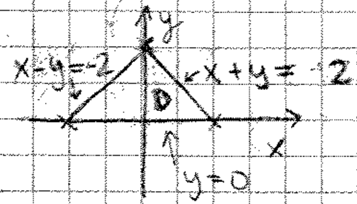
$\therefore \quad x\, g''_{tx} + 2x^2 g''_{tt} + g'_t = x^2 \iff$

$\iff u\left(g''_{vu} - 2u\, g''_{vv}\right) + 2u^2 g''_{vv} + g'_v = u^2 \iff$

$\iff \underline{\underline{u\, g''_{vu} + g'_v = u^2}} \iff \underline{g''_{vu} + \frac{1}{u} g'_v = u}$

(Kan lösas m. hj.a. integrerande faktor.)

6) $I = \iint\limits_D (x^2 - y^2) \sin(x+y)\, dx\, dy$

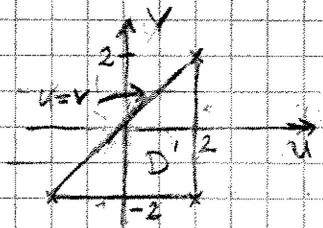


Nya variabler: $\begin{cases} u = x+y \\ v = x-y \end{cases}$

Jacobian:

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \Rightarrow \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}\right| = \left|\frac{1}{-2}\right| = \frac{1}{2}$

$\therefore \quad I = \iint\limits_{D'} uv \sin u \,\frac{du\,dv}{2} =$



$= \frac{1}{2} \int_{-2}^{2}\int_{-2}^{u} v u \sin u \, dv\, du =$

$= \frac{1}{2} \int_{-2}^{2} \left[\frac{v^2}{2}\right]_{-2}^{u} u \sin u \, du =$

$(-2,0) \to (-2,-2)$

$(0,2) \to (2,-2)$

$(2,0) \to (2,2)$

$= \frac{1}{2} \int_{-2}^{2} \frac{(u^2-4)}{2} u \sin u \, du =$

jämn integrand

$= \frac{1}{4} \int_{-2}^{2} (u^3 - 4u) \sin u \, du = \frac{2}{4} \int_0^2 (u^3 - 4u) \sin u \, du =$

$= \frac{1}{2} \left(\left[-(u^3 - 4u) \cos u\right]_0^2 + \int_0^2 (3u^2 - 4) \cos u \, du \right) =$

Partiell integration

$= \frac{1}{2} \left(\left[(3u^2 - 4) \sin u\right]_0^2 - \int_0^2 6u \sin u \, du \right) =$

$= \frac{1}{2} \left(8 \sin 2 - \left(\left[-6u \cos u\right]_0^2 + \int_0^2 6 \cos u \, du \right) \right) =$

$= \frac{1}{2} \left(8 \sin 2 + 12 \cos 2 + \left[6 \sin u\right]_0^2 \right) =$

$= \frac{1}{2} \left(8 \sin 2 + 12 \cos 2 + 6 \sin 2 \right) = \underline{\underline{7 \sin 2 + 6 \cos 2}}$

7) $\vec{AB} = (1-2,\ -1,\ 3-2) = (-1, -1, 1)$

$\therefore\ r(t) = (2-t,\ 1-t,\ 2+t)\ , \quad 0 \le t \le 1$

$r'(t) = (-1,\ -1,\ 1)$

$$\int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t)\, dt = (*)$$

$$F(r(t)) = \big(1-t+2(2-t)(2+t)^2,\ 2-t+1-t+2(1-t)(2+t),\ (1-t)^2+2(2-t)^2(2+t)\big) =$$

$$= \big(1-t+2(4-t^2)(2+t),\ 3-2t+2(2+t-2t-t^2),\ 1-2t+t^2+2(4-t^2)(2-t)\big) =$$

$$= \big(1-t+2(8+4t-2t^2-t^3),\ 7-4t-2t^2,\ 1-2t+t^2+2(8-4t-2t^2+t^3)\big) =$$

$$= \big(17+7t-4t^2-2t^3,\ 7-4t-2t^2,\ 17-10t-3t^2+2t^3\big)$$

$$F(r(t)) \cdot r'(t) = -17-7t+4t^2+2t^3-7+4t+2t^2+17-10t-3t^2+2t^3 =$$

$$= -7-13t+3t^2+4t^3$$

$$\therefore\ (*) = \int_0^1 -7-13t+3t^2+4t^3\, dt = \Big[-7t - \frac{13t^2}{2} + t^3 + t^4\Big]_0^1 =$$

$$= -7 - \frac{13}{2} + 1 + 1 = -\frac{23}{2}$$

Alt.

Sök ϕ s.a. $\nabla \phi = G \iff \begin{cases} \phi'_x = y + 2xz^2 & (1) \\ \phi'_y = x + y + 2yz & (2) \\ \phi'_z = y^2 + 2x^2 z & (3) \end{cases}$

Integrera (1) map. x : $\phi = xy + x^2z^2 + g(y,z)$

Sätt in i (2): $x + g'_y = x + y + 2yz \iff g = y^2 z + \frac{y^2}{2} + h(z)$

Sätt in $\phi = xy + \frac{y^2}{2} + x^2z^2 + y^2 z + h(z)$ i (3):

$2x^2 z + \frac{y^2}{2} + h'_z = y^2 + 2x^2 z \iff h = C$. Välj $C = 0$!

$\therefore\ \phi = xy + \frac{y^2}{2} + x^2 z^2 + y^2 z$

$$\therefore\ \int_C F(x,y) \cdot dr = \phi(1,0,3) - \phi(2,1,2) = 9 - \Big(2 + \frac{1}{2} + 16 + 2\Big)$$

$$= -\frac{23}{2}$$

8) Parametrisering: $\mathbb{r}(u,v) = (u, v, u^2+v^2)$, $u^2+v^2 \leq 1$.

$$\begin{cases} \mathbb{r}'_u = (1, 0, 2u) \\ \mathbb{r}'_v = (0, 1, 2v) \end{cases}$$

$$\mathbb{r}'_u \times \mathbb{r}'_v = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1)$$

Normal riktning utåt ger $\mathbb{n} = (2u, 2v, -1)$.

$$\therefore \Phi = \iint_S \mathbb{F} \cdot d\mathbb{S} = \iint_{u^2+v^2 \leq 1} \mathbb{F}(\mathbb{r}(u,v)) \cdot \mathbb{n} \, du\,dv =$$

$$= \iint_{u^2+v^2 \leq 1} (u, v, (u^2+v^2)^2) \cdot (2u, 2v, -1) \, du\,dv =$$

$$= \iint_{u^2+v^2 \leq 1} 2u^2 + 2v^2 - (u^2+v^2)^2 \, du\,dv = [\text{Polära koord.}] =$$

$$= \int_0^{2\pi}\int_0^1 (2r^2 - r^4)\, r \, dr\, d\theta = 2\pi \int_0^1 2r^3 - r^5 \, dr =$$

$$= 2\pi \left[\frac{r^4}{2} - \frac{r^6}{6} \right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{6} \right) = 2\pi \frac{3-1}{6} = \underline{\underline{\frac{2\pi}{3}}}$$

<u>Alt:</u> <u>Parametrisering</u>: $\mathbb{r}(r,t) = (r\cos t, r\sin t, r^2)$, $0 \leq r \leq 1$, $0 \leq t \leq 2\pi$

$$\begin{cases} \mathbb{r}'_r = (\cos t, \sin t, 2r) \\ \mathbb{r}'_t = (-r\sin t, r\cos t, 0) \end{cases}$$

$$\mathbb{r}'_r \times \mathbb{r}'_t = \begin{vmatrix} \mathbb{i} & \mathbb{j} & \mathbb{k} \\ \cos t & \sin t & 2r \\ -r\sin t & r\cos t & 0 \end{vmatrix} = (-2r^2\cos t, -2r^2\sin t, r\sin^2 t + r\cos^2 t) =$$

$$= (-2r^2\cos t, -2r^2 \sin t, r)$$

Normalriktning utåt ger: $\mathbb{n} = (2r^2\cos t, 2r^2\sin t, -r)$

$$\therefore \Phi = \iint_S \mathbb{F} \cdot d\mathbb{S} = \int_0^{2\pi}\int_0^1 (r\cos t, r\sin t, r^4) \cdot (2r^2\cos t, 2r^2\sin t, -r)\, dr\,dt =$$

$$= \int_0^{2\pi}\int_0^1 2r^3\cos^2 t + 2r^3\sin^2 t - r^5 \, dr\,dt = \int_0^{2\pi}\int_0^1 2r^3 - r^5 \, dr\,dt =$$

$$= 2\pi \left[\frac{r^4}{2} - \frac{r^6}{6} \right]_0^1 = 2\pi \left(\frac{1}{2} - \frac{1}{6} \right) = \underline{\underline{\frac{2\pi}{3}}}$$