Analys B1 Lösningsförslag Tentamen 06 08 26

1) a) Se boken.

b) $r(t) = (3t, 4\cos t, 4\sin t)$

$r'(t) = (3, -4\sin t, 4\cos t)$

$|r'(t)| = \sqrt{9 + 16\sin^2 t + 16\cos^2 t} = \sqrt{25} = 5$

$\therefore$ Längden $= \int_0^{4\pi} 5\,dt = \underline{\underline{20\pi}}$

2) a) $y = x: \quad f(x,x) = \frac{0}{x-1} = 0 \to 0$ då $x \to 1$

$y = 1: \quad f(x,1) = \frac{x^2-1}{x-1} = x+1 \to 2 \neq 0$ då $x \to 1$

$\therefore$ Gränsvärdet existerar ej!

b) $f(3,1) = \frac{9-1}{3-1} = 4$

$f'_x = \frac{2x(x-1) - (x^2-y^2)}{(x-1)^2} = \frac{x^2+y^2-2x}{(x-1)^2} \Rightarrow f'_x(3,1) = \frac{4}{4} = 1$

$f''_{xx} = \frac{(2x-2)(x-1)^2 - 2(x-1)(x^2+y^2-2x)}{(x-1)^4} = 2\,\frac{1-y^2}{(x-1)^3}$

$\Rightarrow f''_{xx}(3,1) = 2\,\frac{0}{4} = 0$

$f''_{xy} = \frac{2y}{(x-1)^2} \Rightarrow f''_{xy}(3,1) = \frac{2}{4} = \frac{1}{2}$

$f'_y = \frac{-2y}{x-1} \Rightarrow f'_y(3,1) = \frac{-2}{2} = -1$

$f''_{yy} = \frac{-2}{x-1} \Rightarrow f''_{yy}(3,1) = \frac{-2}{2} = -1.$

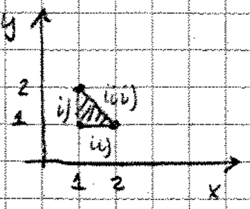
$\therefore P_2(3,1) = 4 + (x-3) - (y-1) + \frac{1}{2}(x-3)(y-1) - \frac{1}{2}(y-1$

3) $\iint_D \sin(\pi x^3) - \sin\left(\frac{\pi y}{x^2}\right) dx\,dy = \int_0^1\int_0^{x^2} \sin(\pi x^3) - \sin\left(\frac{\pi y}{x^2}\right) dy\,dx =$

$= \int_0^1 \left[y\sin(\pi x^3) + \frac{x^2}{\pi}\cos\left(\frac{\pi y}{x^2}\right)\right]_0^{x^2} dx = \int_0^1 x^2\sin(\pi x^3) + \frac{x^2}{\pi}(\cos\pi - \cos 0)$

$= \left[\frac{-1}{3\pi}\cos(\pi x^3) + \frac{x^3}{3\pi}(-2)\right]_0^1 = -\frac{1}{3\pi}(\cos\pi - \cos 0) - \frac{2}{3\pi} = 0$

4.) $f(x,y) = e^{x^2+y^2-3x}$



f är kontinuerlig på slutet och begränsat (kompakt) område, så största och minsta värde existerar.

$$\begin{cases} f'_x = (2x-3)\, e^{x^2+y^2-3x} \\ f'_y = 2y\, e^{x^2+y^2-3x} \end{cases}$$

1) Stationära punkter:

$$\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases} \iff \begin{cases} 2x-3=0 \\ 2y = 0 \end{cases} \iff \begin{cases} x = 3/2 \\ y = 0 \end{cases}$$

Men $(3/2, 0)$ ligger ej i området.

2) Singulära punkter saknas.

3) Randen:

i) $x = 1$, $1 \leq y \leq 2$:

$g(y) = f(1,y) = e^{y^2-2}$, $1 \leq y \leq 2$.

$g' = 2y\, e^{y^2-2}$ $\quad g' = 0 \iff y = 0$.

Men $(1,0)$ ligger ej i området.

Kandidater: $(1,1)$, $(1,2)$

ii) $y = 1$, $1 \leq x \leq 2$:

$h(x) = f(x,1) = e^{x^2-3x+1}$, $1 \leq x \leq 2$

$h' = (2x-3)\, e^{x^2-3x+1}$ $\quad h' = 0 \iff 2x-3=0 \iff x = \frac{3}{2}$

Kandidater: $(\frac{3}{2}, 1)$, $(1,1)$, $(2,1)$

iii) $y = 3-x$ $\quad 1 \leq x \leq 2$

$\varphi(x) = f(x,-x) = e^{x^2+(3-x)^2-3x} = e^{2x^2-9x+9}$

$\varphi'(x) = (4x-9)\, e^{2x^2-3x}$ $\quad \varphi'(x) = 0 \iff 4x-9=0 \iff x = \frac{9}{4}$

Men $(\frac{9}{4}, \frac{3}{4})$ ligger ej i området.

Kandidater: $(2,1)$, $(1,2)$

$f(1,1) = e^{-1}$, $f(2,1) = e^{-1}$, $f(1,2) = e^{2}$, $f(\frac{3}{2}, 1) = e^{-5/4}$

$\therefore$ Största värdet är e^2 och minsta värdet är $e^{-5/4}$.

5) $g'_x = g'_u \frac{\partial u}{\partial x} + g'_v \frac{\partial v}{\partial x} = y g'_u + \frac{g'_v}{y}$

$g''_{xx} = y\left(g''_{uu}\frac{\partial u}{\partial x} + g''_{uv}\frac{\partial v}{\partial x}\right) + \frac{1}{y}\left(g''_{vu}\frac{\partial u}{\partial x} + g''_{vv}\frac{\partial v}{\partial x}\right) =$

$= y\left(g''_{uu}\, y + \frac{1}{y} g''_{uv}\right) + \frac{1}{y}\left(g''_{uv}\, y + g''_{vv}\frac{1}{y}\right) =$

$= y^2 g''_{uu} + 2 g''_{uv} + \frac{1}{y^2} g''_{vv}$

$g'_y = g'_u \frac{\partial u}{\partial y} + g'_v \frac{\partial v}{\partial y} = x g'_u - \frac{x}{y^2} g'_v$

$g''_{yy} = x\left(g''_{uu}\frac{\partial u}{\partial y} + g''_{uv}\frac{\partial v}{\partial y}\right) - \frac{x}{y^2}\left(g''_{vu}\frac{\partial u}{\partial y} + g''_{vv}\frac{\partial v}{\partial y}\right) =$

$= x\left(x g''_{uu} - \frac{x}{y^2} g''_{uv}\right) - \frac{x}{y^2}\left(g''_{uv}\, x - \frac{x}{y^2} g''_{vv}\right) + 2\frac{x}{y^3} g'_v$

$= x^2 g''_{uu} - 2\frac{x^2}{y^2} g''_{uv} + \frac{x^2}{y^4} g''_{vv} + 2\frac{x}{y^3} g'_v$

∴ $x^2 g''_{xx} = y^2 g''_{yy} \iff$

$\iff x^2y^2 g''_{uu} + 2x^2 g''_{uv} + \frac{x^2}{y^2} g''_{vv} =$

$= x^2y^2 g''_{uu} - 2x^2 g''_{uv} + \frac{x^2}{y^2} g''_{vv} + 2\frac{x}{y} g'_v \iff$

$\iff 4x^2 g''_{uv} - 2\frac{x}{y} g'_v = 0 \iff g''_{uv} - \frac{1}{2xy} g'_v = 0 \iff$

$\iff g''_{uv} - \frac{1}{2u} g'_v = 0$

6) Sätt $F(x,y,z) = 2yz^2 - 6x^2y^3 + y^2 z$

$F'_z = 4yz + y^2 \Rightarrow F'_z(-1,2,3) = 24 + 4 = 28 \neq 0$

∴ Implicita funktionssatsen ger att ekvationen definierar z som en funktion av x och y, $z = f(x,y)$, nära punkten $(-1,2,3)$.

$F'_y = 2z^2 - 18x^2y^2 + 2yz \Rightarrow F'_y(-1,2,3) = 18 - 72 + 12 = -42$

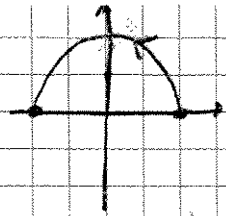
∴ $\frac{\partial f}{\partial y}(-1,2) = -\frac{F'_y(-1,2,3)}{F'_z(-1,2,3)} = -\frac{-42}{28} = \frac{3}{2}$

Alt. Derivera $2yz^2 - 6x^2y^3 + y^2 z = 0$ implicit map. y:

$2z^2 + 4yz z'_y - 18x^2y^2 + 2yz + y^2 z'_y = 0$

$(x,y,z) = (-1,2,3) \Rightarrow 18 + 24 z'_y - 72 + 12 + 4 z'_y = 0 \iff z'_y = \frac{3}{2}$

7) Parametrisering av C:



$$\begin{cases} x = \cos t \\ y = \sin t \end{cases} \quad 0 \le t \le \pi$$

$$\mathbb{r}(t) = (x, y) = (\cos t, \sin t) \Rightarrow \mathbb{r}'(t) = (-\sin t, \cos t)$$

$$\mathbb{F}(\mathbb{r}) = (\cos^2 t - \sin t, \sin t)$$

$$\therefore \int_C \mathbb{F} \cdot d\mathbb{r} = \int_0^{\pi} \mathbb{F}(\mathbb{r}) \cdot \mathbb{r}'(t)\, dt =$$

$$= \int_0^{\pi} (\cos^2 t - \sin t, \sin t) \cdot (-\sin t, \cos t)\, dt =$$

$$= \int_0^{\pi} -\cos^2 t \sin t + \sin^2 t + \sin t \cos t \, dt =$$

$$= \left[\frac{\cos^3 t}{3}\right]_0^{\pi} + \int_0^{\pi} \frac{1 - \cos 2t}{2}\, dt + \left[\frac{\sin^2 t}{2}\right]_0^{\pi} =$$

$$= \frac{(-1)^3 - 1^3}{3} + \left[\frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{\pi} + 0 = -\frac{2}{3} + \frac{\pi}{2} = \frac{\pi}{2} - \frac{2}{3}$$

8) a) $x^2 + y^2 + z^2 = 2 \ \wedge \ z = x^2 + y^2 \iff z^2 + z - 2 = 0 \ \wedge \ z = x^2 + y^2$

$$\iff z = x^2 + y^2 = -\frac{1}{2} (\pm) \sqrt{2 + \frac{1}{4}} = -\frac{1}{2} + \frac{3}{2} = 1.$$

Cylindriska koordinater: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, $0 \le \theta \le 2\pi$, $0 \le r \le 1$, $r^2 \le z \le \sqrt{2 - r^2}$

$$V = \iiint_{\substack{x^2+y^2+z^2 \le 2 \\ z \ge x^2+y^2}} dV = \int_0^{2\pi}\int_0^1\int_{r^2}^{\sqrt{2-r^2}} r \, d\theta\, dr\, dz =$$

$x^2 + y^2 + z^2 = 2$

$\iff z = \pm\sqrt{2 - x^2 - y^2}$

ty $z \ge 0$.

$$= 2\pi \int_0^1 r \left[z\right]_{r^2}^{\sqrt{2-r^2}} dr =$$

$$= 2\pi \int_0^1 r\sqrt{2 - r^2} - r^3 \, dr = 2\pi \left[-\frac{(2-r)^{3/2}}{3} - \frac{r^4}{4}\right]_0^1 =$$

$$= 2\pi \left(-\frac{1}{3} - \frac{1}{4} + \frac{2^{3/2}}{3}\right) = 2\pi\left(\frac{2\sqrt{2}}{3} - \frac{7}{12}\right) = \frac{\pi}{6}(8\sqrt{2} - 7)$$

8b) Sätt $G(x,y,z) = x^2+y^2+z^2-2$

$$\nabla G(x,y,z) = (2x, 2y, 2z)$$

$\therefore$ En normal till ytan är $\mathbf{n} = (x,y,z)$

$$\left|\frac{\mathbf{n}}{n_3}\right| = \frac{\sqrt{x^2+y^2+z^2}}{z} \underset{\substack{\uparrow \\ \text{ty } z\geq 0}}{=} \frac{\sqrt{2}}{\sqrt{2-(x^2+y^2)}}$$

$$\therefore \text{ Arean} = \iint_S dS = \iint_{x^2+y^2\leq 1} \left|\frac{\mathbf{n}}{n_3}\right| dx\,dy = \iint_{x^2+y^2\leq 1} \frac{\sqrt{2}}{\sqrt{2-(x^2+y^2)}}\,dx\,dy =$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{2-r^2}}\,dr\,d\theta = 2\sqrt{2}\pi\left[-\sqrt{2-r^2}\right]_0^1 =$$

$$= 2\sqrt{2}\pi\left(-1+\sqrt{2}\right) = 2\pi\left(2-\sqrt{2}\right)$$

__Alt:__ $\mathbf{r}(u,v) = (u, v, \sqrt{2-u^2-v^2})$

$$\begin{cases} \mathbf{r}'_u = \left(1, 0, \dfrac{-u}{\sqrt{2-u^2-v^2}}\right) \\ \mathbf{r}'_v = \left(0, 1, \dfrac{-v}{\sqrt{2-u^2-v^2}}\right) \end{cases}$$

$$\mathbf{r}'_u \times \mathbf{r}'_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -u/\sqrt{2-u^2-v^2} \\ 0 & 1 & -v/\sqrt{2-u^2-v^2} \end{vmatrix} = \left(\frac{u}{\sqrt{2-u^2-v^2}}, \frac{v}{\sqrt{2-u^2-v^2}}, 1\right)$$

$$|\mathbf{r}'_u \times \mathbf{r}'_v| = \sqrt{\frac{u^2}{2-u^2-v^2}+\frac{v^2}{2-u^2-v^2}+1} = \frac{\sqrt{2}}{\sqrt{2-u^2-v^2}}$$

$$\therefore \text{ Arean} = \iint_{u^2+v^2\leq 1} \frac{\sqrt{2}}{\sqrt{2-u^2-v^2}}\,du\,dv = \sqrt{2}\int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{2-r^2}}\,dr =$$

$$= 2\pi\sqrt{2}\left[-\sqrt{2-r^2}\right]_0^1 = 2\pi\left(2-\sqrt{2}\right)$$

__Alt:__ $\mathbf{r}(u,v) = \sqrt{2}(\sin u\cos v, \sin u \sin v, \cos u)$,

$$\mathbf{r}'_u = \sqrt{2}(\cos u\cos v, \cos u\sin v, -\sin u)$$

$$\mathbf{r}'_v = \sqrt{2}(-\sin u\sin v, \sin u\cos v, 0)$$

$$\mathbf{r}'_u \times \mathbf{r}'_v = \ldots = 2(\sin^2 u\cos v, \sin^2 u\sin v, \cos u\sin u)$$

$$|\mathbf{r}'_u \times \mathbf{r}'_v| = 2\sin u$$

$$\text{Arean} = 2\int_0^{2\pi}\int_0^{\pi/4} \sin u\,du\,dv = 2\cdot 2\pi\left[-\cos u\right]_0^{\pi/4} = 2\cdot 2\pi\left(-\frac{1}{\sqrt{2}}+1\right) = 2\pi\left(2-\sqrt{2}\right)$$